Fall 2020

Variable-Order Fractional Partial Differential Equations: Analysis, Approximation and Inverse Problem

Xiangcheng Zheng

Variable-order fractional partial differential equations: analysis, approximation and inverse problem

by

Xiangcheng Zheng

Bachelor of Engineering
China University of Petroleum, 2016

Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2020

Accepted by:

Hong Wang, Major Professor

Lili Ju, Committee Member

Paula Vasquez, Committee Member

Guiren Wang, Committee Member

Qi Wang, Committee Member

Cheryl L. Addy, Vice Provost and Dean of the Graduate School

Acknowledgments

First and foremost, I would like to express my deepest appreciation for my supervisor Prof. Hong Wang. Prof. Wang provides me not only the detailed and constructive instructions on my study and research, but also the strongest support in all aspects of my doctorate life. Prof. Wang is always available to discuss problems encountered in my research and is enthusiastic and patient to work together with me. The discussions may even last for three hours, during which I gradually realize the way of thinking to resolve the hurdles. Prof. Wang pays much attention on the writing and organization of my papers, which significantly improves my ability of presentation. Furthermore, Prof. Wang offers very strong financial supports, and provides many valuable opportunities of attending conferences, performing presentations and communicating with experts, which grant me a comprehensive improvement. Without the insightful guidance of Prof. Wang, I could not complete this dissertation smoothly.

I would show my sincere thanks to Prof. Vincent Ervin at Clemson University, who provides me professional trainings and detailed instructions on my research, as well as significant supports throughout my doctorate life. In particular, the elegant ideas and techniques he taught me are always impressive. I also sincerely thank Prof. Hongfei Fu at Ocean University of China for the supports and opportunities he provided to improve my academic experiences.

I am extremely grateful to the committee members, including Prof. Lili Ju, Prof. Paula Vasquez, Prof. Guiren Wang and Prof. Qi Wang for their excellent instructions on my courses in the last four and a half years, and insightful guidance and valuable comments and suggestions on my dissertation. I would sincerely thank departmental faculties and staffs, including Prof. Matthew Boylan, Miss. Julia Clark-Spohn, Prof. Maria Girardi, Prof. Xinfeng Liu, Prof. Linyuan Lu, Prof. Yi Sun, Prof. Changhui Tan, Prof. Zhu Wang and Prof. Xiaofeng Yang for their directions and helps in my doctorate life, which significantly improved me in many aspects of study and research.

I extremely appreciate my collaborators, including Prof. Chuanjun Chen, Prof. Jin Cheng, Prof. Pingfei Dai, Prof. Jinhong Jia, Yiqun Li, Dr. Huan Liu, Fangyuan Wang, Zhiwei Yang, Prof. Zhongqiang Zhang and Prof. Zhaojie Zhou, for their elegant ideas, in-depth discussions and valuable comments, suggestions and contributions to our joint papers, which greatly improved the quality of these works.

I would also express my gratitude to my friends at South Carolina, including Shuliang Bai, Pingfei Dai, Siwei Duo, Huadong Gao, Changhong Guo, Xu Guo, Aditya Harish, Yulin Huang, Cuicui Ji, Xiaobo Jing, Hongwei Li, Qi Li, Yiqun Li, Zheng Li, Yixin Lin, Huan Liu, Shuang Liu, Mingyang Pan, Josiah Reiswig, Yuankai Teng, Zhiyu Wang, Zhen Xu, Shaoyun Yi, Xiaofei Yi, Shuai Yuan, Chenfei Zhang, Xinfen Zhang, Zengyan Zhang, Jingyu Zhao, Meng Zhao and Xueping Zhao, for the time that we stay together.

Finally I would like to show my appreciation for my family for their trust and support. I could never come this far without the efforts of my family.

Abstract

Variable-order fractional partial differential equations provide a competitive means in modeling challenging phenomena such as the anomalous diffusion and the memory effects and thus attract widely attentions. However, variable-order fractional models exhibit salient features compared with their constant-order counterparts and introduce mathematical and numerical difficulties that are not common in the context of integer-order and constant-order fractional partial differential equations.

This dissertation intends to carry out a comprehensive investigation on the mathematical analysis and numerical approximations to variable-order fractional derivative problems, including variable-order time-fractional, space-fractional, and space-time fractional partial differential equations, as well as the corresponding inverse problems. Novel techniques are developed to accommodate the impact of the variable fractional order and the proposed mathematical and numerical methods provide potential tools to analyze and compute the variable-order fractional problems.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

The concept of the fractional derivative dates back to Leibniz at 1695 and extends the integer-order derivative to the fractional case. In the past several decades, the nonlocal feature of the fractional operators opens up great opportunities for adequately describing the memory and hereditary properties of physical processes and thus leads to new applications in many fields such as anomalously diffusive transport [69, 106], viscoelastic mechanics [75] and financial mathematics [32, 66, 80]. Consequently, fractional partial differential equations (FPDEs), in which the temporal or/and spatial fractional derivatives are involved in the PDE models, attract increasing attentions [4, 9, 10, 11, 13, 15, 16, 17, 22, 26, 27, 31, 34, 36, 39, 43, 48, 49, 50, 58, 72, 84, 88, 90, 97, 101, 111].

However, FPDEs introduce new mathematical issues that are not common in the context of integer-order PDEs. It was shown in [38, 71, 77, 85] that the first-order time derivative of the solutions to the time-fractional diffusion equations (tFDEs) of order $0 < \alpha < 1$ had a singularity of order $O(t^{\alpha-1})$ near the initial time $t = 0$, which makes the error estimates in the literature that were proved under full regularity assumptions of the true solutions inappropriate. Nevertheless, the singularity of the solutions to the tFDEs at $t = 0$ does not seem to be physically relevant to the diffusive transport processes. Similar issue happens in space-fractional diffusion equations (sFDEs). In [23, 93, 94], the solutions of sFDEs were shown to exhibit singularities near the boundary under the assumptions that the data of the equations are sufficiently smooth.

It is getting clear that the fundamental reason why this phenomenon occurs in tFDEs lies in the incompatibility between the power law decaying tail and the locality of the initial condition at the time $t = 0$. Intuitively, to eliminate the nonphysical singularity of the solutions, as $t \to 0$ the power law decaying tail should switch smoothly to an exponentially decaying tail to account for the impact of locality of the initial condition at the time $t = 0$. That is, a physically relevant tFDE should switch to a classical integer-order diffusion equation near the time $t = 0$, since the power of the heavy tail relates closely to the order of the tFDEs. In other words, variable-order tFDEs, which have a variable-order that approaches to an integer order near the time $t = 0$, provide a physically relevant and feasible fix to the conventional constant-order tFDE models, while capturing the anomalously diffusive transport behavior that the constant-order tFDEs intend to model.

We emphasize that the variable-order tFDEs do not just provide a physically relevant fix of the constant-order tFDEs, but occur in a variety of real applications. In such applications as bioclogging [6], nonconventional hydrocarbon or gas recovery [29], design of shape memory polymer [53], manufacturing of viscoelastic materials [75] and biomaterials in orthopedic implants [99], the structure of porous materials may evolve in time. As the order of a FPDE is related to the fractal dimension of the porous material via the Hurst index [69], these applications may be properly modeled by variable-order tFDEs [53, 60, 87, 112].

Variable-order fractional operators were first proposed and studied in [79], in which they were proved to present salient differences compared with constant-order fractional operators. Many basic properties of the constant-order fractional operators, e.g., the semigroup property (or the index law) of the constant-order fractional integral, do not hold in the variable-order case. Therefore, the impact of the variable fractional orders introduces hurdles that are not encountered in the contest of constant-order cases when carrying out mathematical and numerical analysis to variable-order

FPDEs. For instance, (**i**) commonly-used techniques such as the Laplace transform do not apply to find closed-form solutions of variable-order FPDEs, which makes the analysis of the solution properties of variable-order FPDEs intricate. Numerically, (**ii**) the coefficients of the L1 discretization method, a widely used method when approximating time-fractional derivatives [42, 56, 85], may lose the monotonicity due to the variably memory effects of variable-order fractional operators, which is key in error estimates. In particular, (**iii**) the singularity of the solutions to variable-order FPDEs needs to be characterized and involved when performing error estimates of numerical approximations, instead of imposing artificial smoothness assumptions on the solutions.

There are several literature on the modeling and numerical computations of variable-order fractional FPDEs [20, 25, 37, 86, 87, 104], while the corresponding rigorous mathematical and numerical analyses are far from well-developed. In [92] a piecewise-constant order FPDE was solved analytically on each piece via the explicit formula for constant-order FPDE, with the solutions on the previous pieces and the solution value at the left end of the current piece as the source term and the initial data, respectively. In [41], the well-posedness of a tFDE with a space-dependent variable order was proved. Since the variable order only depends on the spatial variable, the Laplace transform in time could still be used to find the solution representation. In [12], the well-posedness, regularity, and asymptotic behavior of a linear fractional integro-differential equations with a time-dependent order was analyzed via the Laplace transform approach as the definition of the variable-order fractional integral was given in terms of the Laplace transform of the convolution kernel. In [46] the uniqueness of the solutions to a time-fractional Fokker-Planck equation with a space-dependent variable order was proved via the energy method. Nevertheless, systematic investigations on variable-order tFDEs with time-dependent variable orders and variable-order sFDEs are still open problems.

We mainly focus on analyzing variable-order tFDEs with time-dependent variable orders in this dissertzation. There are two different types of variable-order fractional operators, namely, variable-order operators with and without hidden memory (cf. (2.5) and (2.6)), the properties of which are significantly different from each other. Loosely speaking, at each time instant, the orders of the variable-order operators without hidden memory are fixed from the initial time to the current instant, and thus they behave like their constant-order analogues. However, the hidden-memory type operators still have varying orders within that time interval, which makes the corresponding theoretical analysis and numerical approximations more difficult.

Theoretically, there are two methods we follow to analyze the proposed models. One is to employ the spectral decomposition method for the variable-order tFDEs and then convert the component ordinary differential equations (ODEs) into equivalent integral equations, in which the impact of the variable order could be handled such that the analysis can be processed. This method is straightforward to implement and could also be extended to analyze variable-order space-time FDEs [116]/ sFDEs [113]/ fractional wave equations [115]/ fractional stochastic differential equations [117]/ FPDEs with non-singular kernels [119, 118] and variably distributed-order tFDEs [102]. Another method is to consider the variable-order fractional term as part of the source term and then apply the solution representation formula of integer-order PDEs via resolvent operators to find a formal expression of the solutions to variable-order tFDEs for analysis. This method is particularly suitable for problems with space-dependent coefficients or variable orders, or to perform L^p estimates of the solutions.

In the numerical aspect, the L1 coefficients of variable-order tFDEs with hidden-memory variable orders lose monotonicity as those for constant-order tFDEs and even variable-order tFDEs without hidden memory, and the bilinear form of the variable-order sFDEs loses coercivity due to the impact of the variable order. These present hurdles in numerical analysis. We develop novel schemes and techniques to resolve these issues and provide accurate approximations to variable-order fractional derivative problems.

The rest of the dissertation is organized as follows: In Chapter 2 we present preliminaries and refer basic properties of constant-order fractional operators. Then we prove mapping properties of variable-order fractional operators to be used subsequently. In Chapter 3 we study the modeling, well-posedness and regularity of variable-order tFDEs as well as their numerical approximations. Convergence estimate of the proposed scheme is rigorously proved without any artificial smoothness assumption on the solutions. In Chapter 4 we analyze and discretize a different kind of variable-order tFDEs, i.e., tFDEs with hidden-memory variable orders. Due to the lack of the monotonic L1 coefficients, we split each entry to a positive-preserving term added by a high-order perturbation to facilitate the error analysis. In Chapter 5 we mathematically and numerically investigate a tFDE with a space-time dependent variable order via the Laplace transform method and resolvent estimates. In Chapter 6 we analyze a variable-order sFDE and propose an indirect collocation approximation to the model. By proving a generalized Gronwall's inequality, sharp convergence rates are obtained. In Chapter 7 we study inverse problems of determining the variable fractional orders in tFDEs and sFDEs based on observations of the solutions.

Chapter 2

Variable-order fractional operators

In this chapter we introduce notations and spaces, and refer basic properties of constant-order fractional operators. Then we prove mapping properties of variable-order fractional operators to be used in subsequent chapters.

2.1 Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ be the sets of real numbers and positive integers, respectively, $m \in \mathbb{N}\cup\{0\}$ and $0 < \mu < 1 \leq p \leq \infty$. Let $\mathcal{I}$ be a boudned interval and $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) be a simply-connected bounded domain with a piecewise smooth boundary $\partial\Omega$ and convex corners, and $\boldsymbol{x} := (x_1, \cdots, x_d) \in \Omega$. The inner product $(\cdot, \cdot)$ on Ω is given by $(g_1, g_2) := \int_\Omega g_1(\boldsymbol{x}) g_2(\boldsymbol{x}) d\boldsymbol{x}$. Let $L^p(\Omega)$ be the space of the pth power Lebesgue integrable functions on Ω with standard modifications for the case $p = \infty$, and $W^{m,p}(\Omega)$ be the space of functions with derivatives up to order m in $L^p(\Omega)$. Let $H^m(\Omega) := W^{m,2}(\Omega)$ and $H^m_0(\Omega)$ be the completion of $C^\infty_0(\Omega)$, the space of infinitely differentiable functions with compact support in Ω, in $H^m(\Omega)$. For non-integer $s \geq 0$, the fractional Sobolev space $H^s(\Omega)$ is defined by interpolation [3]. All the spaces are equipped with standard norms and could be defined similarly on $\mathcal{I}$ [3].

Let $L_{loc}(\mathcal{I})$, $AC^m(\mathcal{I})$ and $C^m(\mathcal{I})$ be spaces of locally Lebesgue integrable functions, functions with absolutely continuous $(m-1)$-th derivatives, and continuous functions with continuous derivatives up to order m on $\mathcal{I}$ equipped with the norm $\|g\|_{C^m(\mathcal{I})} := \max_{0\leq k\leq m} \sup_{t\in\mathcal{I}} |\partial_t^k g(t)|$, where $\partial_t g(t) = g'(t)$ refers to the first derivative of $g(t)$. In particular, $C(\mathcal{I}) := C^0(\mathcal{I})$. Let $C^\mu(\mathcal{I})$ be the standard Banach space of Hölder continuous functions of order μ equipped with the norm

$$\|g\|_{C^\mu(\mathcal{I})} := \|g\|_{C(\mathcal{I})} + \sup_{t_1,t_2\in\mathcal{I}} \frac{|g(t_2)-g(t_1)|}{|t_2-t_1|^\mu}.$$

Furthermore, for a Banach space $\mathcal{X}$ endowed with the norm $\|\cdot\|_{\mathcal{X}}$, define $C^m(\mathcal{I};\mathcal{X})$ by the space of functions with continuous derivatives up to order m on $\mathcal{I}$ belonging to $\mathcal{X}$ equipped with the norm $\|g\|_{C^m(\mathcal{I};\mathcal{X})} := \max_{0\le k\le m}\sup_{t\in\mathcal{I}} \|\partial_t^k g(\cdot,t)\|_{\mathcal{X}}$, and let $L^p(\mathcal{I};\mathcal{X})$ and $H^1(\mathcal{I};\mathcal{X})$ denote the spaces of functions g such that $\|g\|_{\mathcal{X}} \in L^p(\mathcal{I})$ and $\|g\|_{\mathcal{X}}, \|\partial_t g\|_{\mathcal{X}} \in L^2(\mathcal{I})$, respectively, equipped with standard norms [3].

Let $\mathcal{B} := -\nabla\cdot\big(\boldsymbol{K}(\boldsymbol{x})\nabla\big)$ with $\nabla := (\partial/\partial x_1,\cdots,\partial/\partial x_d)^\top$ and $\boldsymbol{K}(\boldsymbol{x}) := (k_{ij}(\boldsymbol{x}))_{i,j=1}^d$, a symmetric diffusion tensor satisfying $k_{ij}\in C^1(\overline{\Omega})$ and $K_*|\boldsymbol{\zeta}|^2 \le \boldsymbol{\zeta}^\top\boldsymbol{K}\boldsymbol{\zeta} \le K^*|\boldsymbol{\zeta}|^2$ for $\boldsymbol{\zeta}\in\mathbb{R}^d$ and $K^*\ge K_*>0$. Let $\{\lambda_i,\phi_i\}_{i=1}^\infty$ be the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\mathcal{B}\phi_i(\boldsymbol{x}) = \lambda_i\phi_i(\boldsymbol{x}),\ \boldsymbol{x}\in\Omega; \quad \phi_i(\boldsymbol{x}) = 0,\ \boldsymbol{x}\in\partial\Omega. \tag{2.1}$$

It is known that the eigenfunctions $\{\phi_i\}_{i=1}^\infty$ of the Sturm-Liouville problem (2.1) form an orthonormal basis in $L^2(\Omega)$ and the corresponding eigenvalues $\{\lambda_i\}_{i=1}^\infty$ form a positive nondecreasing sequence that tends to ∞ [24]. For any $\gamma\ge 0$ define the Sobolev space $\check{H}^\gamma(\Omega)$ by [3, 77, 91]

$$\check{H}^\gamma(\Omega) := \Big\{v\in L^2(\Omega) : |v|^2_{\check{H}^\gamma} := \big(\mathcal{B}^\gamma v, v\big) = \sum_{i=1}^\infty \lambda_i^\gamma (v,\phi_i)^2 < \infty\Big\}. \tag{2.2}$$

It is known that $\check{H}^\gamma(\Omega)$ is a subspace of $H^\gamma(\Omega)$, and $\check{H}^0(\Omega) = L^2(\Omega)$ and $\check{H}^2(\Omega) = H_0^1(\Omega)\cap H^2(\Omega)$.

Lemma 2.1. *(Generalized Gronwall's inequality [103]) Let* $0\le C_0(t)\in L_{loc}(a,b]$ *and* C_1 *be a non-negative constant. Suppose* $0\le g(t)\in L_{loc}(a,b]$ *satisfies*

$$g(t) \le C_0(t) + C_1\int_a^t g(s)(t-s)^{\gamma-1}ds, \quad \forall t\in(a,b], \quad 0<\gamma<1.$$

Then g *can be bounded by*

$$g(t) \le C_0(t) + \int_a^t \sum_{n=1}^\infty \frac{(C_1\Gamma(\gamma))^n}{\Gamma(n\gamma)}(t-s)^{n\gamma-1}C_0(s)ds, \quad \forall t\in(a,b].$$

In particular, if $C_0(t)$ is non-decreasing, then

$$g(t) \le C_0(t) E_{\gamma,1}\Big(C_1 \Gamma(\gamma)(t-a)^\gamma\Big), \quad \forall t \ \in (a,b],$$

where $E_{p,q}(t)$ represents the two-parameter Mittag-Leffler function defined by [75]

$$E_{p,q}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(pk+q)}, \qquad t \in \mathbb{R}, \quad 0 < p \in \mathbb{R}, \quad q \in \mathbb{R}.$$

In this paper we use Q to denote generic positive constants that may assume different values at different occurrences. Q_i, Q_* and M_i represent fixed constants within each chapter but may be assigned different values in different chapters. For convenience, we may drop the subscript L^2 in $(\cdot,\cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ as well as the notation Ω in the Sobolev spaces and norms, and abbreviate $W^{m,p}(0,T;\mathcal{X})$ as $W^{m,p}(\mathcal{X})$, when no confusion occurs.

2.2 Constant-order fractional operators and their properties

For $0 < \beta < \infty$ and $0 \le a < b < \infty$, the left and right fractional integral operators ${}_aI_x^\beta$ and ${}_xI_b^\beta$ are defined via the Gamma function $\Gamma(\cdot)$ [44, 78]

$$ {}_aI_x^\beta g := \frac{1}{\Gamma(\beta)} \int_a^x \frac{g(s)}{(x-s)^{1-\beta}} ds, \quad {}_xI_b^\beta g := \frac{1}{\Gamma(\beta)} \int_x^b \frac{g(s)}{(s-x)^{1-\beta}} ds, \tag{2.3}$$

based on which we present left and right Riemann-Liouville and Caputo fractional derivative operators of order $0 \le \alpha \notin \mathbb{N}$ (denoted by ${}^R_a\partial_x^\alpha$, ${}^R_x\partial_b^\alpha$, ${}_a\partial_x^\alpha$ and ${}_x\partial_b^\alpha$, respectively) by [44, 78]

$$\begin{aligned} {}^R_a\partial_x^\alpha g &:= \partial_x^n {}_aI_x^{n-\alpha} g, \quad {}^R_x\partial_b^\alpha g := (-\partial_x)^n {}_xI_b^{n-\alpha} g, \\ {}_a\partial_x^\alpha g &:= {}_aI_x^{n-\alpha} \partial_x^n g, \quad {}_x\partial_b^\alpha g := {}_xI_b^{n-\alpha} (-\partial_x)^n g, \end{aligned} \tag{2.4}$$

where $n := [\alpha] + 1$ with $[\alpha] \in (\alpha - 1, \alpha]$ representing the integer part of α.

The following properties of these operators hold [44].

Lemma 2.2. *For $0 \le \alpha \notin \mathbb{N}$, $n = [\alpha] + 1$ and $g \in AC^n[a, b]$, the following relations hold*

$$
{}_a^R\partial_x^\alpha g = \sum_{k=0}^{n-1} \frac{\partial_x^k g(a)(x-a)^{k-\alpha}}{\Gamma(1+k-\alpha)} + {}_a\partial_x^\alpha g,
$$
$$
{}_x^R\partial_b^\alpha g = \sum_{k=0}^{n-1} \frac{(-\partial_x)^k g(b)(b-x)^{k-\alpha}}{\Gamma(1+k-\alpha)} + {}_x\partial_b^\alpha g.
$$

Lemma 2.3. *For $0 < \beta, \beta_1, \beta_2 < \infty$ and $g \in L^1(a, b)$, the following relations hold almost everywhere*

$$
{}_aI_x^{\beta_1}\, {}_aI_x^{\beta_2} g = {}_aI_x^{\beta_1+\beta_2} g, \quad {}_xI_b^{\beta_1}\, {}_xI_b^{\beta_2} g = {}_xI_b^{\beta_1+\beta_2} g,
$$
$$
{}_a^R\partial_x^\beta\, {}_aI_x^\beta g = g, \quad {}_x^R\partial_b^\beta\, {}_xI_b^\beta g = g.
$$

Lemma 2.4. *For $0 < \beta < \infty$, $p, q \ge 1$ and $1/p + 1/q \le 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ if $1/p + 1/q = 1 + \alpha$), it holds for $g_1 \in L^p(a, b)$ and $g_2 \in L^q(a, b)$*

$$
\int_a^b g_1(x)\, {}_aI_x^\beta g_2(x) dx = \int_a^b g_2(x)\, {}_xI_b^\beta g_1(x) dx.
$$

2.3 Variable-order fractional operators and mapping properties

Variable-order fractional operators, in which the constant orders α and β in (2.3)-(2.4) are replaced by functions $n - 1 \le \alpha(x) < n$ for some $n \in \mathbb{N}$ and $0 < \beta(x) < \infty$ on $[a, b]$, are defined analogously [79, 60]

$$
\begin{aligned}
{}_aI_x^{\beta(x)} g &:= \frac{1}{\Gamma(\beta(x))} \int_a^x \frac{g(s)}{(x-s)^{1-\beta(x)}} ds, \\
{}_xI_b^{\beta(x)} g &:= \frac{1}{\Gamma(\beta(x))} \int_x^b \frac{g(s)}{(s-x)^{1-\beta(x)}} ds, \\
{}_a^R\partial_x^{\alpha(x)} g &:= \partial_x^n\, {}_aI_x^{n-\alpha(x)} g, \quad {}_x^R\partial_b^{\alpha(x)} g := (-\partial_x)^n\, {}_xI_b^{n-\alpha(x)} g, \\
{}_a\partial_x^{\alpha(x)} g &:= {}_aI_x^{n-\alpha(x)} \partial_x^n g, \quad {}_x\partial_b^{\alpha(x)} g := {}_xI_b^{n-\alpha(x)} (-\partial_x)^n g.
\end{aligned} \tag{2.5}
$$

Another class of variable-order fractional operators, which are also known as variable-order operators with hidden memory, are given by [86, 87]

$$
\begin{aligned}
{}_a\hat{I}_x^{\beta(x)} g &:= \int_a^x \frac{g(s)}{\Gamma(\beta(s))(x-s)^{1-\beta(s)}} ds, \\
{}_x\hat{I}_b^{\beta(x)} g &:= \int_x^b \frac{g(s)}{\Gamma(\beta(s))(s-x)^{1-\beta(s)}} ds, \\
{}_a^R\hat{\partial}_x^{\alpha(x)} g &:= \partial_x^n\, {}_a\hat{I}_x^{n-\alpha(x)} g, \quad {}_x^R\hat{\partial}_b^{\alpha(x)} g := (-\partial_x)^n\, {}_x\hat{I}_b^{n-\alpha(x)} g, \\
{}_a\hat{\partial}_x^{\alpha(x)} g &:= {}_a\hat{I}_x^{n-\alpha(x)} \partial_x^n g, \quad {}_x\hat{\partial}_b^{\alpha(x)} g := {}_x\hat{I}_b^{n-\alpha(x)} (-\partial_x)^n g.
\end{aligned}
\tag{2.6}
$$

For variable-order fractional operators, properties in Lemmas 2.2-2.4 no longer hold [79, 76], which significantly complicates the mathematical and numerical analysis. Furthermore, the variable order in (2.5) assumes its current value at x for $x \in [a, b]$, while in (2.6) the memory effect depends on the state at s, which exhibits salient differences from (2.5) as we will see in the rest of the dissertation.

The following mapping properties of the variable-order fractional integral operators hold [98, 120, 121].

Theorem 2.5. *Suppose* $\beta \in C^1[a, b]$ *and* $0 < \beta_* \le \beta(x) \le 1$. *Then* ${}_aI_x^{\beta(x)}1$, ${}_a\hat{I}_x^{\beta(x)}1 \in C^{\beta(a)}[a, b]$.

Proof. We prove this theorem by the following two steps.

Step 1: Analysis of ${}_aI_x^{\beta(x)}1$**.** By the definition of ${}_aI_x^{\beta(x)}1$ we obtain

$$
{}_aI_x^{\beta(x)}1 = \frac{(x-a)^{\beta(x)} - (x-x)^{\beta(x)}}{\Gamma(\beta(x))\beta(x)} = \frac{(x-a)^{\beta(x)}}{\Gamma(1+\beta(x))}. \tag{2.7}
$$

As $1/\Gamma(1 + \beta(x))$ is continuously differentiable on $x \in [a, b]$, it suffices to analyze $(x - a)^{\beta(x)}$. With loss of generality, we only consider the case $a \le x_1 < x_2 \le b$ with $x_2 - x_1 < 1$. We begin with $\beta(a) < 1$ and apply the following splitting

$$
(x_2-a)^{\beta(x_2)} - (x_1-a)^{\beta(x_1)} = \Big((x_2-a)^{\beta(x_2)} - (x_2-a)^{\beta(x_1)}\Big) + \Big((x_2-a)^{\beta(x_1)} - (x_1-a)^{\beta(x_1)}\Big).
$$

By the mean value theorem, there exists some $x_1 < \xi < x_2$ such that

$$\begin{aligned} &\left|(x_2-a)^{\beta(x_2)} - (x_2-a)^{\beta(x_1)}\right| \\ &\quad = \left|(x_2-a)^{\beta(\xi)} \ln(x_2-a)\beta'(\xi)\right| |x_2-x_1| \le Q|x_2-x_1|. \end{aligned} \tag{2.8}$$

We first assume $x_1 > a$. For $x_2 - x_1 \le x_1 - a$, again the mean value theorem yields

$$\begin{aligned} &\left|(x_2-a)^{\beta(x_1)} - (x_1-a)^{\beta(x_1)}\right| = \beta(x_1)\left|(\xi-a)^{\beta(x_1)-1}(x_2-x_1)\right| \\ &\le \beta(x_1)(x_1-a)^{\beta(x_1)-1}(x_2-x_1) \\ &= \beta(x_1)(x_1-a)^{\beta(x_1)-\beta(a)}(x_1-a)^{\beta(a)-1}(x_2-x_1) \\ &\le Q(x_1-a)^{\beta(a)-1}(x_2-x_1) \le Q(x_2-x_1)^{\beta(a)-1}(x_2-x_1) = Q(x_2-x_1)^{\beta(a)} \end{aligned} \tag{2.9}$$

where at the second " $\le$ " we have used the fact that

$$\left|(x_1-a)^{\beta(x_1)-\beta(a)}\right| = \left|e^{(\beta(x_1)-\beta(a))\ln(x_1-a)}\right| \le e^{\|\beta\|_{C^1[a,b]}|(x_1-a)\ln(x_1-a)|} \le Q. \tag{2.10}$$

For $x_2 - x_1 > x_1 - a$, we apply the fact

$$y_2^\gamma - y_1^\gamma \le (y_2-y_1)^\gamma, \quad 0 \le y_1 \le y_2, \quad 0 \le \gamma \le 1 \tag{2.11}$$

with $y_1 = x_1 - a$, $y_2 = x_2 - a$ and $\gamma = \beta(x_1)$ to obtain

$$\left|(x_2-a)^{\beta(x_1)} - (x_1-a)^{\beta(x_1)}\right| \le (x_2-x_1)^{\beta(x_1)} \le Q(x_2-x_1)^{\beta(a)}, \tag{2.12}$$

where we have used the estimate

$$\begin{aligned} (x_2-x_1)^{\beta(x_1)-\beta(a)} &= (x_2-x_1)^{\beta'(\xi)(x_1-a)} \le (x_2-x_1)^{-\|\beta\|_{C^1[a,b]}(x_1-a)} \\ &\le (x_1-a)^{-\|\beta\|_{C^1[a,b]}(x_1-a)} \le Q. \end{aligned}$$

We combine (2.8)–(2.12) with triangular inequality to prove the first statement in the lemma for $x_1 > a$. (2.9) and (2.12) hold trivially for $x_1 = a$.

For $\beta(a) = 1$ we differentiate $(x-a)^{\beta(x)}$ with respect to x to get

$$\left[(x-a)^{\beta(x)}\right]' = (x-a)^{\beta(x)}\beta'(x)\ln(x-a) + \beta(x)(x-a)^{\beta(x)-1} \in C(a,b].$$

We take the limit at $x = a^+$ and use $\beta \in C^1[a,b]$ to obtain

$$\lim_{x\to a^+}\left[(x-a)^{\beta(x)}\right]' = \lim_{x\to a^+}(x-a)^{\beta(x)-\beta(a)} = \lim_{x\to a^+} e^{(\beta(x)-\beta(a))\ln(x-a)} = 1.$$

Hence, $(x-a)^{\beta(x)-1} \in C[a,b]$, and so $(x-a)^{\beta(x)} \in C^1[a,b]$.

Step 2: Analysis of ${}_a\hat{I}_x^{\beta(x)}1$. Different from (2.7), we could not evaluate ${}_a\hat{I}_x^{\beta(x)}1$ into a closed form due to the s-dependent of β, cf. (2.6). Thus, we write the kernel in terms of $\beta(x)$ and use integration by parts to get

$$\begin{aligned}
{}_a\hat{I}_x^{\beta(x)}1 &= \int_a^x \frac{ds}{\Gamma(\beta(s))(x-s)^{1-\beta(s)}} = \int_a^x \frac{1}{\Gamma(\beta(s))(x-s)^{\beta(x)-\beta(s)}}\frac{ds}{(x-s)^{1-\beta(x)}}\\
&= -\int_a^x \frac{d(x-s)^{\beta(x)}}{\beta(x)\Gamma(\beta(s))(x-s)^{\beta(x)-\beta(s)}}\\
&= \frac{(x-a)^{\beta(a)}}{\beta(x)\Gamma(\beta(a))} + \int_a^x \frac{(x-s)^{\beta(x)}}{\beta(x)}\partial_s\left[\frac{(x-s)^{\beta(s)-\beta(x)}}{\Gamma(\beta(s))}\right]ds\\
&= \frac{(x-a)^{\beta(a)}}{\beta(x)\Gamma(\beta(a))} + \int_a^x \frac{(x-s)^{\beta(s)}}{\beta(x)}\bigg[-\frac{\Gamma'(\beta(s))\beta'(s)}{\Gamma(\beta(s))^2}\\
&\qquad + \frac{1}{\Gamma(\beta(s))}\left(\beta'(s)\ln(x-s) + \frac{\beta(x)-\beta(s)}{x-s}\right)\bigg]ds.
\end{aligned}$$

As $\beta \in C^1[a,b]$, the integrand on the right-hand side is continuous and so the integral belongs to $C^1[a,b]$. We use (2.11) to conclude that $(x-a)^{\beta(a)} \in C^{\beta(a)}[a,b]$, which completes the proof of this theorem. □

Theorem 2.6. *Suppose* $\beta \in C^1[a,b]$, $0 < \beta_* \le \beta(x) \le 1$ *and* $g \in C^\gamma[a,b]$ *for* $\gamma \ge 0$ *and* $0 < \beta_* + \gamma < 1$. *Then* ${}_aI_x^{\beta(x)}(g - g(a))$, ${}_a\hat{I}_x^{\beta(x)}(g-g(a)) \in C^{\beta_*+\gamma}[a,b]$ *and*

$$\|{}_aI_x^{\beta(x)}(g-g(a))\|_{C^{\beta_*+\gamma}[a,b]} + \|{}_a\hat{I}_x^{\beta(x)}(g-g(a))\|_{C^{\beta_*+\gamma}[a,b]} \le Q\|g\|_{C^\gamma[a,b]}$$

with $Q = Q(a,b,\beta_*,\gamma)$.

Proof. We first estimate ${}_aI_x^{\beta(x)}(g-g(a))$. We assume $a \le x_1 < x_2 \le b$ with $x_2 - x_1 < 1$

and decompose ${}_aI_x^{\beta(x)}(g-g(a))|_{x=x_2} - {}_aI_x^{\beta(x)}(g-g(a))|_{x=x_1}$ as

$$
\begin{aligned}
&\frac{1}{\Gamma(\beta(x_2))}\int_a^{x_2}\frac{g(s)-g(a)}{(x_2-s)^{1-\beta(x_2)}}ds - \frac{1}{\Gamma(\beta(x_1))}\int_a^{x_1}\frac{g(s)-g(a)}{(x_1-s)^{1-\beta(x_1)}}ds \\
&\quad = \Big(\frac{1}{\Gamma(\beta(x_2))} - \frac{1}{\Gamma(\beta(x_1))}\Big)\int_a^{x_1}\Big[g(s)-g(a)\Big](x_1-s)^{\beta(x_1)-1}ds \\
&\qquad + \frac{g(x_1)-g(a)}{\Gamma(\beta(x_2))}\Big[\int_a^{x_2}(x_2-s)^{\beta(x_2)-1}ds - \int_a^{x_1}(x_1-s)^{\beta(x_1)-1}ds\Big] \\
&\qquad + \int_a^{x_1}\frac{\Big[g(s)-g(x_1)\Big]}{\Gamma(\beta(x_2))}\Big[(x_2-s)^{\beta(x_2)-1} - (x_1-s)^{\beta(x_1)-1}\Big]ds \\
&\qquad + \int_{x_1}^{x_2}\frac{\Big[g(s)-g(x_1)\Big]ds}{\Gamma(\beta(x_2))(x_2-s)^{1-\beta(x_2)}} =: I_1+I_2+I_3+I_4.
\end{aligned}
\tag{2.13}
$$

We use $\Gamma \in C^1[\beta_*,1]$, $\beta \in C^1[a,b]$ and $s = x_1+\theta(x_2-x_1)$ (in I_4) to obtain

$$
\begin{aligned}
|I_1| &\le \frac{\|g\|_{L^\infty(a,b)}(x_1-a)^{\beta(x_1)}\Big|\Gamma(\beta(x_2))-\Gamma(\beta(x_1))\Big|}{\Gamma(\beta(x_2))\Gamma(\beta(x_1)+1)} \le Q\|g\|_{L^\infty(a,b)}|x_2-x_1|, \\
|I_4| &\le \frac{Q\|g\|_{C^\gamma[a,b]}}{\Gamma(\beta(x_2))}\int_{x_1}^{x_2}(s-x_1)^\gamma(x_2-s)^{\beta(x_2)-1}ds \\
&= \frac{Q\|g\|_{C^\gamma[a,b]}(x_2-x_1)^{\beta(x_2)+\gamma}B(\gamma+1,\beta(x_2))}{\Gamma(\beta(x_2))} \le Q\|g\|_{C^\gamma[a,b]}(x_2-x_1)^{\beta_*+\gamma}.
\end{aligned}
$$

We bound I_2 similarly by

$$
\begin{aligned}
|I_2| &\le \frac{Q\|g\|_{C^\gamma[a,b]}(x_1-a)^\gamma}{\Gamma(\beta(x_2))}\bigg|\frac{(x_2-a)^{\beta(x_2)}}{\beta(x_2)} - \frac{(x_1-a)^{\beta(x_1)}}{\beta(x_1)}\bigg| \\
&\le \frac{Q\|g\|_{C^\gamma[a,b]}(x_1-a)^\gamma}{\Gamma(\beta(x_2))}\frac{(x_2-a)^{\beta(x_2)}\Big|\beta(x_1)-\beta(x_2)\Big|}{\beta(x_1)\beta(x_2)} \\
&\quad + \frac{Q\|g\|_{C^\gamma[a,b]}(x_1-a)^\gamma}{\Gamma(\beta(x_2))\beta(x_1)}\Big|(x_2-a)^{\beta(x_2)} - (x_1-a)^{\beta(x_1)}\Big| \\
&\le Q\|g\|_{C^\gamma[a,b]}\Big[(x_2-x_1) + (x_1-a)^\gamma|(x_2-a)^{\beta(x_2)} - (x_1-a)^{\beta(x_1)}|\Big].
\end{aligned}
\tag{2.14}
$$

For $x_2 - x_1 \ge x_1 - a$, we use Lemma 2.5 to bound the second term by

$$
(x_1-a)^\gamma\Big|(x_2-a)^{\beta(x_2)} - (x_1-a)^{\beta(x_1)}\Big| \le Q(x_2-x_1)^{\beta(0)+\gamma}.
$$

For $x_2 - x_1 < x_1 - a$, we use the mean-value theorem to obtain

$$
\begin{aligned}
&(x_1-a)^{\gamma}\Big|(x_2-a)^{\beta(x_2)}-(x_1-a)^{\beta(x_1)}\Big| \\
&\quad =(x_1-a)^{\gamma}\Big|(\xi-a)^{\beta(\xi)}\Big(\beta'(\xi)\ln(\xi-a)+\beta(\xi)(\xi-a)^{-1}\Big)\Big|(x_2-x_1) \\
&\quad \le Q(x_2-x_1)+Q(x_1-a)^{\gamma+\beta(\xi)-1}(x_2-x_1)\le Q(x_2-x_1)^{\beta_*+\gamma}.
\end{aligned}
$$

We incorporate the estimates into (2.14) to get $|I_2|\le Q\|g\|_{C^{\gamma}[a,b]}(x_2-x_1)^{\beta_*+\gamma}$.

For $x_2-x_1\ge x_1-a$, we use the substitution $y=x_1-s$ to bound I_3 by

$$
\begin{aligned}
|I_3| &=\Big|\int_0^{x_1-a}\frac{g(x_1-y)-g(x_1)}{\Gamma(\beta(x_2))}\Big[(x_2-x_1+y)^{\beta(x_2)-1}-y^{\beta(x_1)-1}\Big]dy\Big| \\
&\le Q\|g\|_{C^{\gamma}[a,b]}\int_0^{x_2-x_1}y^{\gamma}\Big[(x_2-x_1+y)^{\beta(x_2)-1}+y^{\beta(x_1)-1}\Big]dy \\
&\le Q\|g\|_{C^{\gamma}[a,b]}\int_0^{x_2-x_1}y^{\gamma}\Big[(x_2-x_1)^{\beta(x_2)-1}+y^{\beta(x_1)-1}\Big]dy \\
&\le Q\|g\|_{C^{\gamma}[a,b]}(x_2-x_1)^{\beta_*+\gamma}.
\end{aligned}
\tag{2.15}
$$

Otherwise, we split I_3 as an integral on $[0,x_2-x_1]$ and one on $[x_2-x_1,x_1-a]$

$$
\begin{aligned}
|I_3|\le Q\|g\|_{C^{\gamma}[a,b]}\bigg[&\Big|\int_0^{x_2-x_1}\Big((x_2-x_1+y)^{\beta(x_2)+\gamma-1}-y^{\beta(x_1)+\gamma-1}\Big)dy\Big| \\
&+\Big|\int_{x_2-x_1}^{x_1-a}\Big((x_2-x_1+y)^{\beta(x_2)+\gamma-1}-y^{\beta(x_2)+\gamma-1}\Big)dy\Big| \\
&+\Big|\int_{x_2-x_1}^{x_1-a}\Big(y^{\beta(x_2)+\gamma-1}-y^{\beta(x_1)+\gamma-1}\Big)dy\Big|\bigg]=:Q\|g\|_{C^{\gamma}[a,b]}(I_{31}+I_{32}+I_{33}).
\end{aligned}
$$

We bounded I_{31} in (2.15). We use the mean value theorem to bound I_{33} by

$$
\begin{aligned}
I_{33} &\le \int_{x_2-x_1}^{x_1-a}y^{\beta(x_1)+\gamma-1}\Big|y^{\beta(x_2)-\beta(x_1)}-1\Big|dy \\
&\le Q(x_2-x_1)\int_{x_2-x_1}^{x_1-a}y^{\beta(x_1)+\gamma-1}y^{\xi}|\ln y|dy\le Q(x_2-x_1).
\end{aligned}
$$

Here we have used the following facts: If $\beta(x_2)\ge\beta(x_1)$, then $0\le\xi\le\beta(x_2)-\beta(x_1)$. Then we have $y^{\xi}\le(x_1-a)^{\xi}\le Q$. Otherwise, $\beta(x_2)-\beta(x_1)<\xi<0$. Then we have $y^{\xi}\le(x_2-x_1)^{\xi}\le(x_2-x_1)^{\beta(x_2)-\beta(x_1)}\le Q$ by a similar bound to (2.10).

We use $y = (x_2 - x_1)\theta$ and Taylor expansion on θ to bound I_{32} by

$$
\begin{aligned}
|I_{32}| &\leq \int_{x_2-x_1}^{x_1-a} \Big[y^{\beta(x_2)-1} - (y + x_2 - x_1)^{\beta(x_2)-1}\Big] y^\gamma dy \\
&\leq (x_1 - a)^{\beta(x_2)-\beta_*} \int_{x_2-x_1}^{x_1-a} \Big[y^{\beta_*-1} - (y + x_2 - x_1)^{\beta_*-1}\Big] y^\gamma dy \\
&\leq Q(x_2 - x_1)^{\beta_*+\gamma} \int_1^\infty \Big[\theta^{\beta_*-1} - (1+\theta)^{\beta_*-1}\Big] \theta^\gamma d\theta \\
&\leq Q(x_2 - x_1)^{\beta_*+\gamma} \int_1^\infty \theta^{\beta_*+\gamma-2} d\theta = Q(x_2 - x_1)^{\beta_*+\gamma}.
\end{aligned}
$$

We incorporate all the estimates into (2.13) to finish the proof of ${}_aI_x^{\beta(x)}(g - g(a))$. The estimate of ${}_a\hat{I}_x^{\beta(x)}(g - g(a))$ can be carried out by similar techniques as above (cf. [121]) and thus be omitted. □

Theorem 2.7. *Suppose $\beta \in C^1[a,b]$, $0 < \beta_* \leq \beta(x) \leq 1$ and $g \in C^\gamma[a,b]$ for $\gamma \geq 0$ and $\beta_* + \gamma > 1$. Then ${}_aI_x^{\beta(x)}(g - g(a))$, ${}_a\hat{I}_x^{\beta(x)}(g - g(a)) \in C^1[a,b]$ and*

$$
\|{}_aI_x^{\beta(x)}(g - g(a))\|_{C^1[a,b]} + \|{}_a\hat{I}_x^{\beta(x)}(g - g(a))\|_{C^1[a,b]} \leq Q\|g\|_{C^\gamma[a,b]}
$$

with $Q = Q(a, b, \beta_*, \gamma)$.

Proof. For $g \in C^\gamma[a,b]$ and $0 < \sigma \ll 1$, let $\tilde{g}(s) := g(s) - g(a)$. Then the function

$$
g_\sigma(x) := \int_a^{x-\sigma} \tilde{g}(s)(x - s)^{\beta(x)-1} ds \in C^1[a + \sigma, b] \tag{2.16}
$$

is differentiable with respect to x and the derivative $g'_\sigma(x)$ can be bounded by

$$
\begin{aligned}
\big|g'_\sigma(x)\big| = &\ \Bigg| \int_a^{x-\sigma} \big(\tilde{g}(s) - \tilde{g}(x)\big)\Big[\frac{\beta'(x)\ln(x-s)}{(x-s)^{1-\beta(x)}} + \frac{\beta(x) - 1}{(x-s)^{2-\beta(x)}}\Big] ds \\
&+ \frac{\tilde{g}(x)}{(x-a)^{1-\beta(x)}} + \tilde{g}(x) \int_a^{x-\sigma} \frac{\beta'(x)\ln(x-s)}{(x-s)^{1-\beta(x)}} ds + \frac{\tilde{g}(x-\sigma) - \tilde{g}(x)}{\sigma^{1-\beta(x)}} \Bigg| \\
\leq &\ Q\|g\|_{C^\gamma[a,b]} \Bigg[\int_a^{x-\sigma} (x-s)^\gamma \Big[\frac{|\ln(x-s)|}{(x-s)^{1-\beta(x)}} + \frac{1}{(x-s)^{2-\beta(x)}}\Big] ds \\
&+ (x-a)^{\beta(x)+\gamma-1} + \int_a^{x-\sigma} \frac{|\ln(x-s)|}{(x-s)^{1-\beta(x)}} ds + \sigma^{\beta(x)+\gamma-1} \Bigg] \\
\leq &\ Q\|g\|_{C^\gamma[a,b]} \Bigg[\int_a^{x-\sigma} (x-s)^{\beta(x)+\gamma-2} ds + 1 \Bigg] \leq Q\|g\|_{C^\gamma[a,b]}, \quad x \in [a+\sigma, b].
\end{aligned}
$$

Consequently, its limiting function

$$\begin{aligned}\psi(x) \quad &:= \lim_{\sigma\to 0^+} g'_\sigma(x) = \frac{\tilde{g}(x)}{(x-a)^{1-\beta(x)}} + \tilde{g}(x)\int_a^x \frac{\beta'(x)\ln(x-s)}{(x-s)^{1-\beta(x)}}ds \\ &+ \int_a^x (\tilde{g}(s)-\tilde{g}(x))\Big[\frac{\beta'(x)\ln(x-s)}{(x-s)^{1-\beta(x)}} + \frac{\beta(x)-1}{(x-s)^{2-\beta(x)}}\Big]ds\end{aligned}$$

is continuous on $[a,b]$ and is bounded by

$$\Big|\psi(x)\Big| \le Q\|g\|_{C^\gamma[a,b]}, \quad x\in[a,b].$$

By Lebesgue bounded convergence theorem,

$$\int_a^x \psi(s)ds = \lim_{\sigma\to 0^+}\int_{a+\sigma}^x g'_\sigma(s)ds = \lim_{\sigma\to 0^+}\Big[g_\sigma(x) - g_\sigma(a+\sigma)\Big]. \tag{2.17}$$

By (2.16)–(2.17) and the definition of ${}_aI_x^{\beta(x)}(g-g(a))$, we conclude that

$${}_aI_x^{\beta(x)}(g-g(a)) = \frac{1}{\Gamma(\beta(x))}\int_a^x \psi(s)ds \in C^1[a,b].$$

The estimate of ${}_a\hat{I}_x^{\beta(x)}(g-g(a))$ can be carried out by similar techniques as above (cf. [121]) and thus be omitted. □

Chapter 3

Variable-order time-fractional PDEs

In this chapter we study the well-posedness and regularity of the following variable-order tFDE model in multiple space dimensions, which was studied in [28] and analyzed in [96]

$$\begin{aligned} &\partial_t u + k\, {}_0\partial_t^{\alpha(t)} u + \mathcal{B}u = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T]; \\ &u(\boldsymbol{x},0) = u_0(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega; \quad u(\boldsymbol{x},t) = 0,\ (\boldsymbol{x},t) \in \partial\Omega \times [0,T]. \end{aligned} \tag{3.1}$$

Here $k \geq 0$ represents a partial fraction. Then we prove an optimal-order error estimate of the corresponding fully-discretized finite element scheme without any artificial smoothness assumption on the solutions. Numerical experiments are carried out to support the theoretical findings.

3.1 Modeling issues

The tFDE

$${}_0\partial_t^{\alpha} u - K\Delta u = f(\boldsymbol{x},t), \quad 0 < \alpha < 1, \tag{3.2}$$

was derived via a continuous time random walk (CMRW) under the assumption that the mean waiting time has a power-law decaying tail [69, 70]. This explains why tFDE (3.2) accurately describes the power-law decaying behavior of the subdiffusive transport, which attracts extensive research [18, 30, 45, 47, 55, 52, 64, 65, 73, 74]. However, tFDE (3.2) exhibits nonphysical initial weak singularity as the first-order time derivative of its solution behaves as $O(t^{\alpha-1})$ near the time $t = 0$ [63, 77, 85].

This makes the error estimates in the literature that were proved under full regularity assumptions of the true solutions inappropriate.

The reason is that tFDE (3.2) was derived as the diffusion limit of a CTRW in the phase space as the number of particle jumps tends to infinity [69, 70], and so holds for relatively large time $t > 0$, rather than all the way up to the time $t = 0$ as often assumed in the literature. This explains why tFDE (3.2) exhibits nonphysical initial weak singularity.

The mobile-immobile tFDE model

$$\partial_t u + k\, {}_0\partial_t^{\alpha} u - K\Delta u = f(\boldsymbol{x}, t), \quad 0 < \alpha < 1 \tag{3.3}$$

was presented in [81] to improve the modeling of subdiffusive transport of solutes in heterogeneous porous media. In this context a large amount of particles may get absorbed to the media and then get released at later time. Consequently, the travel time of the adsorbed particles gets significantly longer and deviates from that of the particles in the bulk phase that undergo a Brownian motion [125]. Thus, the adsorbed particles have power-law decaying tails and undergo subdiffusive transport. In (3.3) the u_t term models the Fickian diffusive transport of the particles in the bulk fluid phase that consist of $1/(1+k)$ portion of the total solute mass for some $k \geq 0$, and the $k\, {}_0\partial_t^{\alpha} u$ term models the subdiffusive transport of the absorbed particles that consist of $k/(1 + k)$ portion of the total solute mass.

In the left figure of Figure 3.1 we show representative plots of the mean square displacement (MSD) $\langle x(t)^2 \rangle$ for the one-dimensional tFDE (3.2) and the mobile-immobile tFDE (3.3) on the space-time domain $(x, t) \in (-10, 10) \times [0, T]$ with $T = 100$, incorporated with the diffusivity coefficient $K = 0.01$, the initial value $u_0(x) = e^{-x^2/(2\times 0.01^2)}/(\sqrt{2\pi} \times 0.01)$ and different values of k. We arrive at the following observations: The mobile-immobile tFDE (3.3) with $k = 0$ models the Fickian diffusive transport of the solute while (3.3) with $k \to \infty$ models the purely subdiffusive transport of the solute. (3.3) with $k > 0$ models a combination of a Fickian

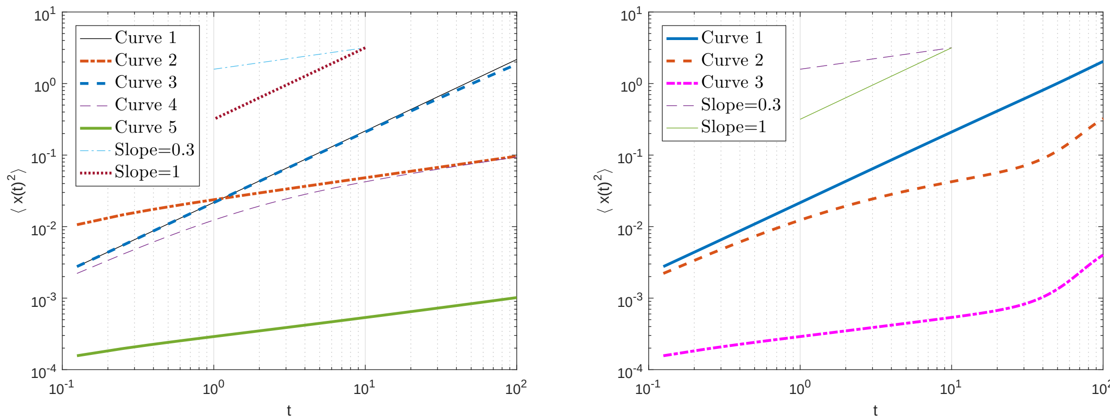

Figure 3.1 Log-log plots of the MSD $\langle x(t)^2 \rangle$ for integer-order and constant-order tFDEs (left) and variable-order tFDEs (right). Left: the integer-order diffusion equation (curve 1), tFDE (3.2) with $\alpha = 0.3$ (curve 2), the mobile-immobile tFDE (3.3) with $\alpha = 0.3$ and $k = 0.01$ (curve 3), 1 (curve 4) and 100 (curve 5), respectively (curves 3-5); Right: the variable-order tFDE (3.1) with $\alpha(t)$ given by (3.33) with $\alpha(0) = 0.3$, $\alpha(T) = 0.7$ and $k = 0.01$ (curve 1), $k = 1$ (curve 2) and $k = 100$ (curve 3), respectively [112, FIG. 1].

diffusive transport and a subdiffusive transport weighted by k. Moreover, in all these cases the integral of the solutions u, i.e., the total solute mass, equal to the initial mass. Hence, tFDE (3.3) can be viewed as a physically relevant extension of tFDE (3.2) to the entire interval including the initial time $t = 0$ [57, 81].

It was reported in the literature that in many applications the mean-square displacement curves have clear slope changes, which cannot be modeled accurately by constant-order tFDEs and requires variable-order tFDE modeling (cf. e.g. [87] and the references therein). In the right figure of Figure 3.1 we present representative plots of the mean square displacement $\langle x(t)^2 \rangle$ generated by the variable-order generalization (3.1) of mobile-immobile tFDE (3.3) with the parameters chosen as before. We observe from Figure 3.1 that variable-order tFDEs can model more complicated anomalously diffusive transport processes, and naturally eliminate the nonphysical singular behavior of the solutions that occurs in constant-order tFDEs at time $t = 0$ [96].

So far, numerical methods of variable-order tFDEs were derived or analyzed under the assumption that the true solutions are smooth [87, 100, 105, 107, 126]. However, there are very limited results on the well-posedness and smoothing properties of variable-order tFDEs, as well as rigorous numerical analysis based on the true regularity of the solutions. In the rest of this chapter, we focus on the mathematical and numerical analysis of model (3.1) based on the following assumptions on the variable order $\alpha(t)$:

Assumption A $\alpha \in C^1[0,T]$ and $0 \le \alpha(t) \le \alpha^* < 1$ for some $0 < \alpha^* < 1$.

3.2 Well-posedness and solution regularity

We follow the integral equation approach in [96, 98] to prove the well-posedness and solution regularity of the variable-order tFDE (3.1).

3.2.1 Well-posedness of an integral equation

We first study the following integral equation, which is motivated by (3.16) below

$$v(t) = -k\,{}_0I_t^{1-\alpha(t)}v + k\,\lambda\,[{}_0I_t^{1-\alpha(t)}v] * e^{-\lambda t} + g(t) - \lambda\, g * e^{-\lambda t} - w_0\lambda e^{-\lambda t}, \tag{3.4}$$

where $*$ denotes the convolution on $[0,t]$, $\lambda > 0$ and $g(t)$ and w_0 are given data.

Lemma 3.1. *Suppose the Assumption A hold and $g \in C[0,T]$, then equation (3.4) has a unique solution $v \in C[0,T]$ and*

$$\|v\|_{C[0,T]} \le Q_0 M_0, \tag{3.5}$$

where $M_0 := \lambda|w_0| + \|g\|_{C[0,T]}$ and $Q_0 = Q_0(\alpha^*, k, T)$.

Proof. We define a sequence of approximations $\{v_n\}_{n=0}^{\infty}$ on $[0,T]$ by

$$\begin{aligned} v_0(t) &:= g(t) - \lambda\, g * e^{-\lambda t} - w_0\lambda e^{-\lambda t}, \\ v_n(t) &:= -k\,{}_0I_t^{1-\alpha(t)}v_{n-1} + k\,\lambda\left[{}_0I_t^{1-\alpha(t)}v_{n-1}\right] * e^{-\lambda t} + v_0(t),\ n \ge 1. \end{aligned} \tag{3.6}$$

It is clear that $\|v_0\|_{C[0,T]} \leq Q_* M_0$ for some $Q_* > 0$. We subtract $v_{n+1}(t)$ from $v_n(t)$ for $n \geq 0$ and apply the estimates $(t-s)^{\alpha^*-\alpha} \leq \max\{1, T\}$ and

$$\begin{aligned}\Big|k\,{}_0I_t^{1-\alpha(t)}(v_n - v_{n-1})\Big| &\leq k \int_0^t \frac{|v_n(s) - v_{n-1}(s)|}{\Gamma(1-\alpha(t))(t-s)^{\alpha(t)}} ds \\ &= k \int_0^t \frac{\Gamma(1-\alpha^*)(t-s)^{\alpha^*-\alpha(t)}}{\Gamma(1-\alpha(t))} \frac{|v_n(s) - v_{n-1}(s)|}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}} ds \\ &\leq Q \int_0^t \frac{|v_n(s) - v_{n-1}(s)|}{\Gamma(1-\alpha^*)(t-s)^{\alpha^*}} ds = Q\,{}_0I_t^{1-\alpha^*}|v_n - v_{n-1}|\end{aligned} \tag{3.7}$$

to conclude that for $t \in [0, T]$

$$\begin{aligned}\Big|v_{n+1}(t) - v_n(t)\Big| &= \Big|k\,{}_0I_t^{1-\alpha(t)}(v_n - v_{n-1}) - k\,\lambda\left[{}_0I_t^{1-\alpha(t)}(v_n - v_{n-1})\right] * e^{-\lambda t}\Big| \\ &\leq Q\,{}_0I_t^{1-\alpha^*}|v_n - v_{n-1}| + Q\lambda\left[{}_0I_t^{1-\alpha^*}|v_n - v_{n-1}|\right] * e^{-\lambda t}.\end{aligned} \tag{3.8}$$

Here $v_{-1} := 0$. We plug the bound for $\|v_0\|_{C^1[0,T]}$ into (3.8) with $n = 0$ and apply

$${}_0I_t^{1-\alpha^*}|v_0| = \frac{1}{\Gamma(1-\alpha^*)} \int_0^t \frac{|v_0(s)|}{(t-s)^{\alpha^*}} = \frac{Q_* M_0 t^{1-\alpha^*}}{\Gamma(2-\alpha^*)}$$

to obtain

$$\begin{aligned}\Big|v_1(t) - v_0(t)\Big| \quad &\leq Q\Big({}_0I_t^{1-\alpha^*}|v_0| + \lambda[{}_0I_t^{1-\alpha^*}|v_0|] * e^{-\lambda t}\Big) \\ &\leq \frac{QQ_* M_0 t^{1-\alpha^*}}{\Gamma(2-\alpha^*)}\Big(1 + \lambda * e^{-\lambda t}\Big) \leq \frac{Q_* M_0 (2Q) t^{1-\alpha^*}}{\Gamma((1-\alpha^*)+1)}.\end{aligned}$$

Assume that for some $n \geq 1$,

$$\Big|v_n(t) - v_{n-1}(t)\Big| \leq \frac{Q_* M_0 (2Q)^n t^{n(1-\alpha^*)}}{\Gamma(n(1-\alpha^*)+1)}, \quad t \in [0, T]. \tag{3.9}$$

Then by (3.8) we obtain

$$\begin{aligned}&\Big|v_{n+1}(t) - v_n(t)\Big| \\ &\leq \frac{Q_* M_0 (2Q)^{n+1}}{2\Gamma(n(1-\alpha^*)+1)}\Big({}_0I_t^{1-\alpha^*} t^{n(1-\alpha^*)} + \lambda\,[{}_0I_t^{1-\alpha^*} t^{n(1-\alpha^*)}] * e^{-\lambda t}\Big) \\ &= \frac{Q_* M_0 (2Q)^{n+1}}{2\Gamma((n+1)(1-\alpha^*)+1)}\Big(t^{(n+1)(1-\alpha^*)} + \lambda\, t^{(n+1)(1-\alpha^*)} * e^{-\lambda t}\Big) \\ &\leq \frac{Q_* M_0 (2Q)^{n+1} t^{(n+1)(1-\alpha^*)}}{2\Gamma((n+1)(1-\alpha^*)+1)}\Big(1 + \lambda\, 1 * e^{-\lambda t}\Big) \leq \frac{Q_* M_0 (2Q)^{n+1} t^{(n+1)(1-\alpha^*)}}{\Gamma((n+1)(1-\alpha^*)+1)}.\end{aligned}$$

As a result, (3.9) holds for any $n \in \mathbb{N}$ by mathematical induction. Noting that the series defined by the right-hand side converges to the Mittag-Leffler function [75]

$$\sum_{j=0}^{\infty} \frac{Q_* M_0 (2Q)^j t^{j(1-\alpha^*)}}{\Gamma(j(1-\alpha^*)+1)} = Q_* M_0 E_{1-\alpha^*,1}\Big(2Qt^{1-\alpha^*}\Big) < \infty, \quad t \in [0,T].$$

The uniform limit v of the left side series $[0,T]$

$$v(t) := \lim_{n\to\infty} v_n(t) = \sum_{n=1}^{\infty} \Big(v_n(t) - v_{n-1}(t)\Big) + v_0(t)$$

satisfies (3.5). We take the limit on both sides of the second equation in (3.6) and use the expression of $v_0(t)$ to conclude that v solves (3.4). Since $v_0 \in C[0,T]$, we apply Theorems 2.5–2.6 and (3.6) to conclude inductively that $v_n \in C[0,T]$ and so $v \in C[0,T]$. Let $\bar{v} \in C[0,T]$ be another solution to (3.4) and it is clear that $\varepsilon(t) := v(t) - \bar{v}(t)$ satisfies

$$\varepsilon(t) = -k\, {}_0I_t^{1-\alpha(t)}\varepsilon + k\,\lambda[{}_0I_t^{1-\alpha(t)}\varepsilon] * e^{-\lambda t}. \tag{3.10}$$

We then apply

$$\int_0^t |\varepsilon(s)|ds = \int_0^t (t-s)^{\alpha^*} \frac{|\varepsilon(s)|}{(t-s)^{\alpha^*}} ds \le \max\{1,T\} \int_0^t \frac{|\varepsilon(s)|}{(t-s)^{\alpha^*}} ds,$$

and the similar techniques in (3.7) to (3.10) to bound $\varepsilon(t)$ by

$$\begin{aligned} |\varepsilon(t)| &\le Q_0 I_t^{1-\alpha^*}|\varepsilon| + Q\lambda \int_0^t |\varepsilon(s)| \int_s^t \frac{1}{(\theta-s)^{\alpha^*}} d\theta ds \\ &\le Q\int_0^t \frac{|\varepsilon(s)|ds}{(t-s)^{\alpha^*}} + Q\lambda \int_0^t |\varepsilon(s)|ds \le Q(1+\lambda)\int_0^t \frac{|\varepsilon(s)|ds}{(t-s)^{\alpha^*}}. \end{aligned}$$

We apply the generalized Gronwall's inequality in Lemma 2.1 to conclude that $\varepsilon(t) \equiv 0$. Hence, the integral equation (3.4) has a unique solution $v \in C[0,T]$ which satisfies the stability estimate (3.5). □

3.2.2 Well-posedness and solution regularity

We prove well-posedness of model (3.1) in the following theorem.

Theorem 3.2. *If Assumption A holds, $u_0 \in \check{H}^{r+2}$ and $f \in H^\kappa(\check{H}^r)$ with $\kappa > 1/2$ and $r > d/2$, then problem (3.1) has a unique solution $u \in C^1([0,T];\check{H}^r)$ and*

$$\|u\|_{C^1([0,T];\check{H}^s)} \le Q_1\Big(\|u_0\|_{\check{H}^{s+2}} + \|f\|_{H^\kappa(\check{H}^s)}\Big), \quad 0 \le s \le r \tag{3.11}$$

where $Q_1 = Q_1(\alpha^*, k, T, \kappa)$.

Proof. We express u and f in (3.1) in terms of $\{\phi_i\}_{i=1}^\infty$ [77, 85]

$$\begin{aligned} u(\boldsymbol{x},t) &= \sum_{i=1}^\infty u_i(t)\phi_i(\boldsymbol{x}), \quad u_i(t) := \Big(u(\cdot,t),\phi_i\Big), \quad t \in [0,T], \\ f(\boldsymbol{x},t) &= \sum_{i=1}^\infty f_i(t)\phi_i(\boldsymbol{x}), \quad f_i(t) := \Big(f(\cdot,t),\phi_i\Big), \quad t \in [0,T]. \end{aligned} \tag{3.12}$$

By plugging (3.12) into (3.1) and using (2.1), we obtain

$$\sum_{i=1}^\infty \Big[u_i'(t) + k\,{}_0\partial_t^{\alpha(t)} u_i(t) + \lambda_i u_i(t) - f_i(t)\Big]\phi_i(\boldsymbol{x}) = 0. \tag{3.13}$$

Hence, u is a solution to problem (3.1) if and only if $\{u_i\}_{i=1}^\infty$ satisfy the following fractional ordinary differential equations (fODEs)

$$\begin{aligned} u_i'(t) + k\,{}_0\partial_t^{\alpha(t)} u_i(t) + \lambda_i u_i(t) &= f_i(t), \quad t \in (0,T], \\ u_i(0) = u_{0,i} &:= (u_0,\phi_i), \quad i = 1,2,\cdots. \end{aligned} \tag{3.14}$$

We integrate the fODEs multiplied by $e^{\lambda_i t}$ to obtain

$$u_i(t) = -\Big(k\,{}_0I_t^{1-\alpha(t)} u_i' - f_i\Big) * e^{-\lambda_i t} + u_{0,i}e^{-\lambda_i t}. \tag{3.15}$$

We further differentiate (3.15) to obtain an integral equation with respect to $u_i'(t) =: v(t)$

$$v(t) = -k\,{}_0I_t^{1-\alpha(t)} v + k\,\lambda_i\,{}_0I_t^{1-\alpha(t)} v * e^{-\lambda_i t} + f_i - \lambda_i\, f_i * e^{-\lambda_i t} - \lambda_i u_{0,i} e^{-\lambda_i t}, \tag{3.16}$$

which is exactly the equation (3.4) with $w_0 = u_{0,i}$, $g = f_i$, and $\lambda = \lambda_i$.

We apply Lemma 3.1 to conclude that equation (3.16) has a unique solution $v \in C[0,T]$ and (3.5) holds. Hence,

$$u_i(t) := u_{0,i} + \int_0^t v(s)ds \in C^1[0,T]$$

solves (3.15) and so (3.14), and the derivations from (3.14) to (3.16) are justified. The uniqueness of the C^1 solutions to (3.14) follows from (3.15) and (3.16).

For any $k, n \in \mathbb{N}$, we use Sobolev embedding and Lemma 3.1 to conclude that $S_n(\boldsymbol{x}, t) := \sum_{i=1}^{n} u_i(t)\phi_i(\boldsymbol{x})$ satisfies that for $n \to \infty$

$$\begin{aligned}\left\|S'_{n+k} - S'_n\right\|^2_{C([0,T];C(\bar\Omega))} &\le Q\Big\|\sum_{i=n+1}^{n+k} u'_i(t)\phi_i(\boldsymbol{x})\Big\|^2_{C([0,T];H^r)}\\ &\le Q\sum_{i=n+1}^{n+k}\lambda_i^r\|u_i\|^2_{C^1[0,T]} \le Q\sum_{i=n+1}^{n+k}\lambda_i^r\Big(\lambda_i^2 u_{0,i}^2 + \|f_i\|^2_{C[0,T]}\Big) \to 0.\end{aligned}$$

Hence, the interchange of the differentiation with the summation in (3.13) is justified, from which we conclude that u defined in (3.12) belongs to $C^1(\check{H}^2)$ and satisfies problem (3.1). We use Lemma 3.1 to obtain for $0 \le s \le r$

$$\begin{aligned}\left\|\partial_t u\right\|^2_{C([0,T];\check{H}^s)} &\le Q\sum_{i=1}^{\infty}\lambda_i^s\|u_i\|^2_{C^1[0,T]} \le Q\sum_{i=1}^{\infty}\lambda_i^s\Big(\lambda_i^2 u_{0,i}^2 + \|f_i\|^2_{C[0,T]}\Big)\\ &= Q\Big(\|u_0\|^2_{\check{H}^{s+2}} + \|f\|^2_{H^\kappa(\check{H}^s)}\Big),\end{aligned}$$

which, together with $u = \int_0^t \partial_t u(\boldsymbol{x}, s)ds + u_0$, yields (3.11).

Finally, let $\bar{u} \in C^1(\check{H}^r)$ be another solution to problem (3.1). Then $u - \bar{u}$ satisfies the homogeneous analogue of (3.1) with the homogeneous initial condition. Then (3.11) yields $u - \bar{u} \equiv 0$, which completes the proof. $\square$

In what follows, we investigate the regularity of $\partial_{tt}u$, which will be used later.

Theorem 3.3. *Suppose the Assumption A holds, $u_0 \in \check{H}^4$ and $f \in H^\kappa(\check{H}^2) \cap H^{1+\kappa}(L^2)$ for some $\kappa > 1/2$. If $\alpha(0) > 0$, then $\partial_{tt}u \in C((0,T];L^2)$ and*

$$\|\partial_{tt}u\| \le Q_2 t^{-\alpha(0)}\Big(\|u_0\|_{\check{H}^4} + \|f\|_{H^\kappa(\check{H}^2)} + \|f\|_{H^{1+\kappa}(L^2)}\Big), \quad 0 < t \le T.$$

If $\alpha(0) = 0$, then $\partial_{tt}u \in C([0,T];L^2)$ and

$$\|\partial_{tt}u\| \le Q_2\Big(\|u_0\|_{\check{H}^4} + \|f\|_{H^\kappa(\check{H}^2)} + \|f\|_{H^{1+\kappa}(L^2)}\Big), \quad 0 \le t \le T.$$

Here $Q_2 = Q_2(\alpha^*, \|\alpha\|_{C^1[0,T]}, k, T, \kappa)$.

Proof. We begin with $\alpha(0) > 0$. We multiply equation (3.16) by t and use $t = s + (t-s)$ to split ${}_0I_t^{1-\alpha(t)}(tv)$ to get the following equation in terms of $tv(t)$

$$\begin{aligned} tv = -k\,{}_0I_t^{1-\alpha(t)}(tv) - k\int_0^t \frac{(t-s)^{1-\alpha(t)}v(s)\,ds}{\Gamma(1-\alpha(t))} \\ + k\lambda_i t\,{}_0I_t^{1-\alpha(t)} v * e^{-\lambda_i t} + tf_i - \lambda_i t f_i * e^{-\lambda_i t} - u_{0,i}\lambda_i t e^{-\lambda_i t}. \end{aligned} \tag{3.17}$$

Since $v \in C[0,T]$ by Lemma 3.1, all but the first terms on the right-hand side are in $C^1[0,T]$. Let $m \in \mathbb{N}^+$ be such that $m(1-\alpha^*) < 1$ and $(m+1)(1-\alpha^*) > 1$ (if $m\alpha^* = 1$, we slightly increase the value of α^*). Then we apply Theorem 2.6 to (3.17) with $tv \in C[0,T]$ to conclude that $tv \in C^{1-\alpha^*}[0,T]$. We repeat the procedure m times to conclude that $tv \in C^{m(1-\alpha^*)}[0,T]$. As $m(1-\alpha^*)+1-\alpha^* > 1$, we apply Theorem 2.7 to (3.17) with $tv \in C^{m(1-\alpha^*)}[0,T]$ to deduce that ${}_0I_t^{1-\alpha(t)}(tv) \in C^1[0,T]$ and thus $tv \in C^1[0,T]$. Thus, v is differentiable for $t \in (0,T]$ and so u_i has a second-order derivative for $t \in (0,T]$.

To derive a stability estimate for v' on $(0,T]$, we differentiate (3.16) to obtain

$$v'(t) = -k\partial_t\,{}_0I_t^{1-\alpha(t)} v + R, \tag{3.18}$$

Here R denotes the derivative of all but the first terms on the right-hand side of (3.16) that

$$R := k\lambda_i {}_0I_t^{1-\alpha(t)} v - k\lambda_i^2 {}_0I_t^{1-\alpha(t)} v * e^{-\lambda_i t} + f_i' - \lambda_i f_i + \lambda_i^2 f_i * e^{-\lambda_i t} + \lambda_i^2 e^{-\lambda_i t} u_{0,i}.$$

We apply integration by parts to obtain

$$\begin{aligned} \Big|\partial_t\,{}_0I_t^{1-\alpha(t)} v\Big| &= \left|\partial_t\Big(\frac{t^{1-\alpha(t)}v(0)}{\Gamma(2-\alpha(t))} + \int_0^t \frac{(t-s)^{1-\alpha(t)}}{\Gamma(2-\alpha(t))} v'(s)ds\Big)\right| \\ &\le Qt^{-\alpha(t)}|v(0)| + Q\int_0^t \frac{|v'(s)|}{(t-s)^{\alpha^*}} ds, \end{aligned}$$

which, together with

$$t^{-\alpha(t)} = t^{-\alpha(0)} t^{\alpha(0)-\alpha(t)} = e^{(\alpha(0)-\alpha(t))\ln t} t^{-\alpha(0)} \le Qt^{-\alpha(0)},$$

and Lemma 3.1, implies

$$\left|k\partial_t\, {}_0I_t^{1-\alpha(t)}v\right| \leq Q\int_0^t \frac{|v'(s)|ds}{(t-s)^{\alpha^*}}ds + \Big(\lambda_i|u_{0,i}| + \|f_i\|_{C[0,T]}\Big)t^{-\alpha(0)}.$$

The terms in R can be simply bounded by $QM_1 := Q\Big(\lambda_i^2|u_{0,i}|+\lambda_i\|f_i\|_{C[0,T]}+\|f_i'\|_{C[0,T]}\Big)$ and we incorporate the preceding estimates into (3.18) to get

$$|v'(t)| \leq Q\int_0^t \frac{|v'(s)|ds}{(t-s)^{\alpha^*}} + QM_1t^{-\alpha(0)}, \quad t \in (0,T]. \tag{3.19}$$

We use Lemma 2.1 to bound $|v'|$ by

$$\begin{aligned}|v'| &\leq QM_1t^{-\alpha(0)} + QM_1\sum_{n=1}^{\infty}\frac{\big(Q\Gamma(\alpha^*)\big)^n}{\Gamma(n\alpha^*)}\int_0^t (t-s)^{n\alpha^*-1}s^{-\alpha(0)}ds \\ &= QM_1t^{-\alpha(0)}\Big(1+\Gamma(1-\alpha(0))\sum_{n=1}^{\infty}\frac{(Q\Gamma(\alpha^*)t^{\alpha^*})^n}{\Gamma(n\alpha^*+1-\alpha(0))}\Big) \leq QM_1t^{-\alpha(0)},\end{aligned}$$

and then use this estimate to arrive at the following stability estimate of $\|\partial_{tt}u\|$

$$\begin{aligned}\|\partial_{tt}u\|^2 &= \sum_{i=1}^{\infty}|u_i''|^2 \leq Qt^{-2\alpha(0)}\left(\sum_{i=1}^{\infty}\lambda_i^4|u_{0,i}|^2 + \sum_{i=1}^{\infty}\Big(\lambda_i^2\|f_i\|^2_{H^\kappa(0,T)} + \|f_i\|^2_{H^{1+\kappa}(0,T)}\Big)\right) \\ &\leq Qt^{-2\alpha(0)}\left(\|u_0\|^2_{\check{H}^4} + \|f\|^2_{H^\kappa(\check{H}^2)} + \|f\|^2_{H^{1+\kappa}(L^2)}\right).\end{aligned}$$

This proves the first estimate in Theorem 3.3. The proof of the second estimate can be carried out similarly and thus be omitted. □

3.3 Discretization and error estimate

In this section we follow [112] to present and analyze a fully-discrete finite element approximation of model (3.1). Define a uniform temporal partition on $[0,T]$ by $t_n := Tn/N$ for $0 \leq n \leq N$ with the stepsize $\tau = T/N$. Define a quasi-uniform partition of Ω with mesh diameter h and let S_h be the space of continuous and piecewise linear functions on this partition. Let I be the identity operator and $\Pi_h : H_0^1(\Omega) \to S_h(\Omega)$ be the Ritz projection operator defined by

$$\Big(\boldsymbol{K}(\cdot)\nabla(g-\Pi_hg), \nabla\chi\Big) = 0, \quad \forall\chi \in S_h, \text{ for } g \in H_0^1(\Omega). \tag{3.20}$$

The following approximation property holds [91]

$$\Big\|(I-\Pi_h)g\Big\| \le Qh^2\|g\|_{H^2}, \quad \forall\, g\in H^2(\Omega)\cap H_0^1(\Omega). \tag{3.21}$$

Let $u_n := u(\boldsymbol{x}, t_n)$. We discretize $\partial_t u$ and ${}_0\partial_t^{\alpha(t)}u$ at $t=t_n$ $(1\le n\le N)$ by

$$\begin{aligned}
\partial_t u(\boldsymbol{x},t_n) &= \delta_\tau u_n + E_n := \frac{u_n-u_{n-1}}{\tau} + \frac{1}{\tau}\int_{t_{n-1}}^{t_n}\partial_{tt}u(\boldsymbol{x},t)(t-t_{n-1})dt,\\
{}_0\partial_t^{\alpha(t_n)}u(\boldsymbol{x},t_n) &= \delta_\tau^{\alpha(t_n)}u_n + R_n = \frac{1}{\Gamma(1-\alpha(t_n))}\\
&\quad\times\sum_{k=1}^{n}\left[\int_{t_{k-1}}^{t_k}\frac{\delta_\tau u_k}{(t_n-t)^{\alpha(t_n)}}dt + \int_{t_{k-1}}^{t_k}\frac{\partial_t u-\delta_\tau u_k}{(t_n-t)^{\alpha(t_n)}}dt\right]
\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}
\delta_\tau^{\alpha(t_n)}u(\boldsymbol{x},t_n) &:= \sum_{k=1}^{n}\left[\frac{\Big(t_n-t_{k-1}\Big)^{1-\alpha(t_n)}-\Big(t_n-t_k\Big)^{1-\alpha(t_n)}}{\Gamma(2-\alpha(t_n))}\right]\delta_\tau u_k = \sum_{k=1}^{n}b_{n,k}\Big(u_k-u_{k-1}\Big),\\
R_n &:= \sum_{k=1}^{n}R_{n,k} := \frac{1}{\Gamma(1-\alpha(t_n))}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{\partial_t u-\delta_\tau u_k}{(t_n-t)^{\alpha(t_n)}}dt\\
&= \frac{1}{\Gamma(1-\alpha(t_n))}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{1}{\tau(t_n-t)^{\alpha(t_n)}}\left[\int_{t_{k-1}}^{t_k}\int_s^t\partial_{\theta\theta}u(\boldsymbol{x},\theta)d\theta ds\right]dt,
\end{aligned} \tag{3.23}$$

$$b_{n,k} := \frac{(t_n-t_{k-1})^{1-\alpha(t_n)}-(t_n-t_k)^{1-\alpha(t_n)}}{\Gamma(2-\alpha(t_n))\tau}, \quad 1\le k\le n\le N. \tag{3.24}$$

$b_{n,k}$ has the following properties [85]

$$\begin{cases} b_{n,n} > b_{n,n-1} > \cdots > b_{n,k} > \dots b_{n,1} > 0,\\ (1-\alpha(t_n))(t_n-t_{k-1})^{-\alpha(t_n)} \le b_{n,k} \le (1-\alpha(t_n))(t_n-t_k)^{-\alpha(t_n)}, \quad k<n. \end{cases} \tag{3.25}$$

We plug (3.22) into (3.1), and integrate the resulting equation multiplied by $\chi\in H_0^1(\Omega)$ on Ω to get a weak formulation for problem (3.1) for any $\chi\in H_0^1(\Omega)$ for $n=1,\cdots,N$

$$(\delta_\tau u_n,\chi)+\Big(\boldsymbol{K}(\cdot)\nabla u_n,\nabla\chi\Big) = -k(\delta_\tau^{\alpha(t_n)}u_n,\chi)+(f(\cdot,t_n),\chi)-\Big(kR_n+E_n,\chi\Big). \tag{3.26}$$

We drop the local truncation error term on the right-hand side to obtain a finite element scheme for (3.1): find $U_n\in S_h$ with $U_0:=\Pi_h u_0$ such that for $n=1,\cdots,N$

$$(\delta_\tau U_n,\chi)+\Big(\boldsymbol{K}(\cdot)\nabla U_n,\nabla\chi\Big) = -k(\delta_\tau^{\alpha(t_n)}U_n,\chi)+(f(\cdot,t_n),\chi), \quad \forall\chi\in S_h. \tag{3.27}$$

3.3.1 Estimates of local truncation errors

In this subsection we prove the following two lemmas.

Lemma 3.4. *Suppose the Assumption A holds,* $\alpha \in C^1[0,T]$ *and* $u_0 \in \check{H}^4$, $f \in H^1(\check{H}^2) \cap H^2(L^2)$. *Then the following time-step wise estimates hold*

$$\|E_n\| \le Q\hat{Q}_0 n^{-\alpha(0)} N^{\alpha(0)-1}, \quad \|R_n\| \le Q\hat{Q}_0 n^{-\alpha^*} N^{\alpha^*-1}, \quad 1 \le n \le N \tag{3.28}$$

with $\hat{Q}_0 := \Big(\|u_0\|_{\check{H}^4} + \|f\|_{H^1(\check{H}^2)} + \|f\|_{H^2(L^2)}\Big)$.

Proof. We begin with $\alpha(0) > 0$. We use the mean-value theorem to get for $n \ge 2$

$$t_n^{1-\alpha(0)} - t_{n-1}^{1-\alpha(0)} \le (1-\alpha(0)) t_{n-1}^{-\alpha(0)} \tau \le Q\Big((n-1)/N\Big)^{-\alpha(0)} \tau \le Q\Big(n/N\Big)^{-\alpha(0)} \tau. \tag{3.29}$$

We apply (3.29) and Theorem 3.3 to bound E_n in (3.22) by the following to get the first estimate in (3.28)

$$\begin{aligned}
\Big\|E_n\Big\| &\le \frac{Q\hat{Q}_0}{\tau} \int_{t_{n-1}}^{t_n} t^{-\alpha(0)}(t - t_{n-1})dt \le Q\hat{Q}_0 \int_{t_{n-1}}^{t_n} t^{-\alpha(0)} dt \\
&= \begin{cases} Q\hat{Q}_0 \tau^{1-\alpha(0)} = Q\hat{Q}_0 N^{\alpha(0)-1}, & n = 1; \\ Q\hat{Q}_0 \big(t_n^{1-\alpha(0)} - t_{n-1}^{1-\alpha(0)}\big) \le Q\hat{Q}_0 n^{-\alpha(0)} N^{\alpha(0)-1}, & n \ge 2. \end{cases}
\end{aligned}$$

We apply Theorem 3.2, which guarantees the boundedness of $\|u\|_{C^1([0,T];L^2)}$, the first equality in (3.23), and the estimates $(t_n - t)^{-\alpha(t_n)} = (t_n - t)^{-\alpha^*}(t_n - t)^{\alpha^* - \alpha(t_n)} \le \max\{1, T\}(t_n - t)^{-\alpha^*}$ and (3.29) to bound $R_{n,1}$ in (3.23) by

$$\begin{aligned}
\Big\|R_{n,1}\Big\| &\le Q\Bigg\| \int_0^{t_1} (t_n - t)^{-\alpha^*} \bigg[\Big|u_t(\cdot, t)\Big| + \frac{1}{\tau}\int_0^{t_1} \Big|u_t(\cdot, s)\Big| ds\bigg] dt \Bigg\| \\
&\le Q\hat{Q}_0 \int_0^{t_1} (t_n - t)^{-\alpha^*} dt \le \begin{cases} Q\hat{Q}_0 \tau^{1-\alpha^*}, & n = 1, \\ Q\hat{Q}_0 (t_n - t_1)^{-\alpha^*} \tau \le Q\hat{Q}_0 \Big(\dfrac{n}{N}\Big)^{-\alpha^*} \tau, & n \ge 2. \end{cases}
\end{aligned}$$

We use the second equality in (3.23) and Theorem 3.3 to bound $R_{n,n}$ for $n > 1$ by

$$\begin{aligned}
\Big\|R_{n,n}\Big\| &\le Q\hat{Q}_0 t_{n-1}^{-\alpha(0)} \tau \int_{t_{n-1}}^{t_n} (t_n - t)^{-\alpha(t_n)} dt \le Q\hat{Q}_0 t_{n-1}^{-\alpha(0)} \tau^{2-\alpha^*} \\
&\le \frac{QQ_0 n^{-\alpha(0)}}{N^{-\alpha(0)}} \frac{1}{N^{2-\alpha^*}} \le \frac{Q\hat{Q}_0}{n^{2-\alpha^*}} \Big(\frac{n}{N}\Big)^{2-\alpha(0)-\alpha^*}.
\end{aligned}$$

We bound the rest of R_n in (3.23) for $n \geq 3$ in the following two parts

$$\begin{aligned}\left\|\sum_{k=\lceil n/2\rceil+1}^{n-1} R_{n,k}\right\| &\leq Q\hat{Q}_0 \sum_{k=\lceil n/2\rceil+1}^{n-1} t_{k-1}^{-\alpha(0)}\tau \int_{t_{k-1}}^{t_k} (t_n-t)^{-\alpha^*}dt \\ &\leq Q\hat{Q}_0 t_{\lceil n/2\rceil}^{-\alpha(0)}\tau \int_{t_{\lceil n/2\rceil}}^{t_{n-1}} (t_n-t)^{-\alpha^*}dt \leq Q\hat{Q}_0 t_n^{-\alpha(0)}\tau\, t_n^{1-\alpha^*} \\ &\leq Q\hat{Q}_0\Big(\frac{n}{N}\Big)^{-\alpha(0)}\frac{1}{N}\Big(\frac{n}{N}\Big)^{1-\alpha^*} \leq \frac{Q\hat{Q}_0}{n}\Big(\frac{n}{N}\Big)^{2-\alpha^*-\alpha(0)},\end{aligned}$$

$$\begin{aligned}\left\|\sum_{k=2}^{\lceil n/2\rceil} R_{n,k}\right\| &\leq Q\hat{Q}_0 \sum_{k=2}^{\lceil n/2\rceil} t_{k-1}^{-\alpha(0)}\tau \int_{t_{k-1}}^{t_k} (t_n-t)^{-\alpha^*}dt \leq Q\hat{Q}_0 \sum_{k=2}^{\lceil n/2\rceil} t_k^{-\alpha(0)}\tau^2(t_n-t_k)^{-\alpha^*} \\ &\leq Q\hat{Q}_0 \sum_{k=2}^{\lceil n/2\rceil} t_k^{-\alpha(0)}\tau^2 t_n^{-\alpha^*} \leq Q\hat{Q}_0 \sum_{k=2}^{\lceil n/2\rceil} \frac{k^{-\alpha(0)}n^{-\alpha^*}}{N^{2-\alpha(0)-\alpha^*}} \leq \frac{Q\hat{Q}_0}{n}\Big(\frac{n}{N}\Big)^{2-\alpha^*-\alpha(0)}.\end{aligned}$$

We incorporate the preceding estimates to obtain the second estimate in (3.28) and thus complete the proof of the lemma. □

Lemma 3.5. *Suppose the Assumption A holds and* $u_0 \in \check{H}^4$, $f \in H^1(\check{H}^2)$. *Then the following estimates for* $\eta_n := \big(I - \Pi_h\big)u_n$ *hold with* $\hat{Q}_1 := \|u_0\|_{\check{H}^4} + \|f\|_{H^1(\check{H}^2)}$

$$\big\|\delta_\tau \eta_n\big\|_{\hat{L}^\infty(0,T;L^2)} := \max_{1\leq n\leq N} \|\delta_\tau\eta_n\| \leq Q\hat{Q}_1 h^2, \quad \|\delta_\tau^{\alpha(t_n)}\eta_n\|_{\hat{L}^\infty(0,T;L^2)} \leq Q\hat{Q}_1 h^2.$$

Proof. We use (3.21) to bound $\big\|\delta_\tau\eta_n\big\|$ for $r \geq 1$ by

$$\big\|\delta_\tau\eta_n\big\| = \frac{1}{\tau}\left\|\int_{t_{n-1}}^{t_n}(I-\Pi_h)\partial_t u dt\right\| \leq Qh^2\|u\|_{C^1([0,T];H^2)} \leq Q\hat{Q}_1 h^2.$$

We then use (3.23) and (3.24) to get

$$\big\|\delta_\tau^{\alpha(t_n)}\eta_n\big\| = \left\|\sum_{k=1}^n b_{n,k}(I-\Pi_h)\int_{t_{n-1}}^{t_n} u_t dt\right\| \leq Qh^2\|u\|_{C^1([0,T];H^2)}\sum_{k=1}^n b_{n,k}\tau \leq Q\hat{Q}_1 h^2.$$

We thus finish the proof of the lemma. □

3.3.2 Optimal-order error estimate of finite element scheme (3.27)

We prove the error estimate without any artificial regularity assumption of the true solution.

Theorem 3.6. *Suppose the Assumption A hold, $u_0 \in \check{H}^4$ and $f \in H^1(\check{H}^2) \cap H^2(L^2)$. Then there is a positive constant $Q = Q(k, T, \alpha^*, \|\alpha\|_{C^1[0,T]})$ such that an optimal-order error estimate holds for finite element scheme* (3.27) *to model* (3.1)

$$\|U - u\|_{\hat{L}^\infty(0,T;L^2)} \leq Q\Big(\|u_0\|_{\check{H}^4} + \|f\|_{H^1(\check{H}^2)} + \|f\|_{H^2(L^2)}\Big)\Big(\tau + h^2\Big).$$

Proof. We split the error $u_n - U_n = \xi_n + \eta_n$ where $\xi_n := \Pi_h u_n - U_n$. The estimate of η_n is given by (3.21) so we remain to bound ξ_n. We subtract (3.27) from (3.26) with $\chi = \xi_n$ and apply (3.20) to obtain the following error equation in terms of ξ_n

$$(\delta_\tau \xi_n, \xi_n) + \Big(\boldsymbol{K}\nabla\xi_n, \nabla\xi_n\Big) = -k\,(\delta_\tau^{\alpha(t_n)}\xi_n, \xi_n) - \Big(k[R_n - \delta_\tau^{\alpha(t_n)}\eta_n] + E_n - \delta_\tau\eta_n, \xi_n\Big). \tag{3.30}$$

We use $\xi_0 := U_0 - \Pi_h u_0 = 0$ to rearrange $\delta_\tau^{\alpha(t_n)}\xi_n$ by

$$\delta_\tau^{\alpha(t_n)}\xi_n = b_{n,n}\xi_n - \sum_{k=1}^{n-1}\Big(b_{n,k+1} - b_{n,k}\Big)\xi_k$$

and reformulate (3.30) as

$$\begin{aligned}&(\xi_n, \xi_n) + \tau\Big(\boldsymbol{K}\nabla\xi_n, \nabla\xi_n\Big) + \tau k b_{n,n}(\xi_n, \xi_n)\\ &\quad = (\xi_{n-1}, \xi_n) + \tau k\sum_{k=1}^{n-1}\Big(b_{n,k+1} - b_{n,k}\Big)(\xi_k, \xi_n) - \tau\Big(k\Big[R_n - \delta_\tau^{\alpha(t_n)}\eta_n\Big] + E_n - \delta_\tau\eta_n, \xi_n\Big).\end{aligned}$$

We use (3.24)-(3.25) to obtain the following

$$\Big(1 + \tau k b_{n,n}\Big)\|\xi_n\| \leq \|\xi_{n-1}\| + \tau k\sum_{k=1}^{n-1}\Big(b_{n,k+1} - b_{n,k}\Big)\|\xi_k\| + \tau G_n, \tag{3.31}$$

where

$$G_n := k\|R_n\| + k\|\delta_\tau^{\alpha(t_n)}\eta_n\| + \|E_n\| + \|\delta_\tau\eta_n\|.$$

It is clear from (3.31) that $\|\xi_1\| \leq \tau G_1$. Assume

$$\|\xi_m\| \leq \tau\sum_{j=1}^{m} G_j, \quad 2 \leq m \leq n-1. \tag{3.32}$$

Plugging (3.32) with $2 \le m \le n-1$ into (3.31) yields

$$
\begin{aligned}
(1+\tau k b_{n,n})\|\xi_n\| &\le \|\xi_{n-1}\| + \tau k \sum_{k=1}^{n-1}\Big(b_{n,k+1}-b_{n,k}\Big)\|\xi_k\| + \tau G_n \\
&\le \tau \sum_{j=1}^{n} G_j + k\tau^2 \sum_{k=1}^{n-1}\Big(b_{n,k+1}-b_{n,k}\Big)\sum_{j=1}^{k} G_j \\
&\le \Big(1+k\tau(b_{n,n}-b_{n,1})\Big)\tau \sum_{j=1}^{n} G_j \le (1+k\tau b_{n,n})\tau \sum_{j=1}^{n} G_j.
\end{aligned}
$$

Thus, (3.32) holds for $m=n$ and so for any $m \ge 2$ by mathematical induction.

We remain to bound the right-hand side of (3.32) for any $1 \le m \le N$. We use Lemmas 3.4 and 3.5 to conclude that

$$
\begin{aligned}
&\tau \sum_{n=1}^{N}(\|E_n\| + \|R_n\|) \le Q\hat{Q}_0 \tau \sum_{n=1}^{N}(n^{-\alpha(0)}N^{\alpha(0)-1} + n^{-\alpha^*}N^{\alpha^*-1}) \le Q\hat{Q}_0\tau, \\
&\tau \sum_{n=1}^{N}(\|\delta_\tau \eta_n\| + \|\delta_\tau^{\alpha(t_n)}\eta_n\|) \le Q\hat{Q}_1 h^2 \sum_{n=1}^{N}\tau = Q\hat{Q}_1 h^2.
\end{aligned}
$$

with $\hat{Q}_0$ and $\hat{Q}_1$ introduced in Lemmas 3.4-3.5. We incorporate these estimates into (3.32) to complete the proof. □

3.4 Numerical experiments

We numerically investigate the regularity of the solutions to the variable-order tFDE (3.1) and its dependence on the behavior of the variable order $\alpha(t)$, as well as the convergence behavior of the finite element scheme (3.27) [112].

3.4.1 Behavior of the solutions to variable-order tFDE (3.1)

Let $\Omega = (0,1)^3$, the time interval $[0,T] = [0,1]$, $k=1$, $\boldsymbol{K} = 0.001\boldsymbol{I}$ with $\boldsymbol{I}$ being the identity matrix of order three, $u_0(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$, $f=0$, and the variable order $\alpha(t)$ is given by

$$
\alpha(t) = \alpha(T) + (\alpha(0)-\alpha(T))\Big(1 - \frac{t}{T} - \frac{\sin(2\pi(1-t/T))}{2\pi}\Big). \qquad (3.33)
$$

We present the numerical solutions $U_n(1/2,1/2,z)$ to the variable-order tFDE (3.1) in Figure 3.2 for the three cases: $(\alpha(0),\alpha(1)) =$ (i) $(0,0.2)$; (ii) $(0.4,0.6)$; (iii) $(0.7,0.9)$.

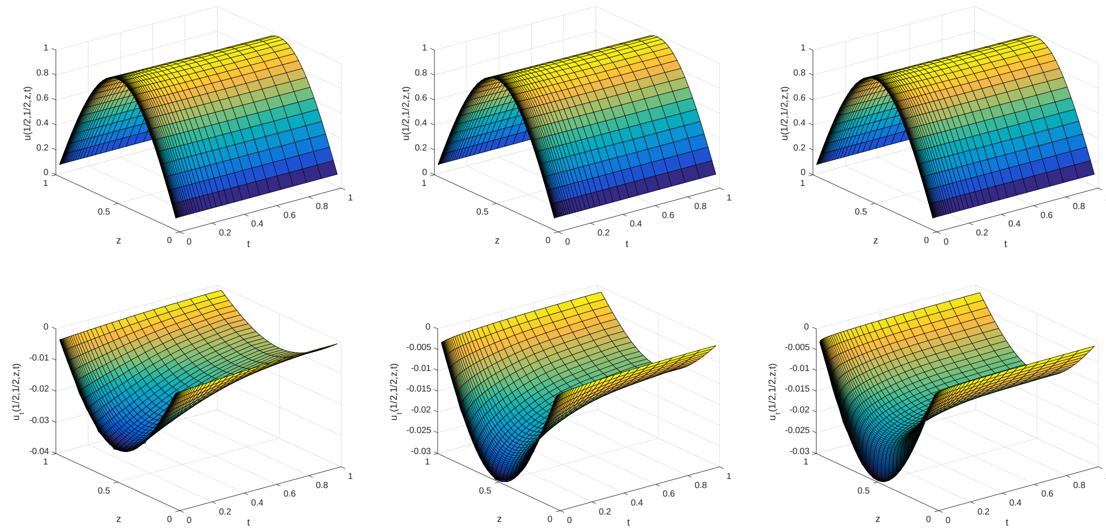

Figure 3.2 Plots of the finite element solutions $U_n(1/2, 1/2, z)$ (the first row) and the first-order time different quotients $\delta_\tau U_n(1/2, 1/2, z)$ of the numerical solutions $U_n(1/2, 1/2, z)$ (the second row) to model (3.1) for the cases (i) left, (ii) middle, and (iii) right [112, FIG. 2 and FIG. 3].

In the numerical experiments we use a uniform spatial partition of a mesh size $h = 1/32$. To better capture the singularity of the solutions near the initial time, we use a graded temporal mesh $t_n = T(n/N)^r$ of $N = 64$ and $r = 2/(1-0.7) \approx 6.7$, which is determined by the most singular case of $\alpha(0) = 0.7$ [112]. We observe that the solutions are smooth in all cases. The time derivative of the solutions in case (i) is smooth near the initial time $t = 0$, while exhibits singularities near the initial time $t = 0$ for cases (ii) and (iii), and the singularities get stronger as $\alpha(0)$ increases. These observations numerically justify the analysis in Theorems 3.2-3.3.

3.4.2 Convergence of the finite element scheme

We investigate the temporal convergence behavior of the finite element scheme to the variable-order tFDE (3.1).

Example 1 Let $\Omega = (0, 1)$, $[0, T] = [0, 1]$, $k(t) = 1$, $\boldsymbol{K} := K = 0.001$, $f = 0$, and let $\alpha(t)$ be given by (3.33). We use the numerical solutions $\hat{U}$ under a uniform spatial partition of $h = 1/32$ and a uniform temporal partition of $\tau = 2^{-10}$ as the reference solutions. We measure the convergence rate κ of the numerical approximations by $\|U - \hat{U}\|_{\hat{L}^\infty(0,T;L^2)} \leq Q\tau^\kappa$. We present the numerical results in Table 3.1 and observe the first-order convergence rates, which confirms the error estimates in Theorem 3.6.

Table 3.1 Temporal accuracy of finite element scheme under $h = 1/32$ and different $(\alpha(0), \alpha(1))$ [112, Tables 1-3].

τ	$(0, 0.4)$	κ	$(0.4, 0.6)$	κ	$(0, 0.7)$	κ	$(0.7, 0.9)$	κ
1/8	5.63E-04		4.64E-04		5.66E-04		3.25E-04	
1/16	2.79E-04	1.02	2.37E-04	0.97	2.80E-04	1.02	1.75E-04	0.90
1/24	1.82E-04	1.05	1.58E-04	1.00	1.83E-04	1.04	1.20E-04	0.93
1/32	1.33E-04	1.08	1.18E-04	1.01	1.35E-04	1.07	9.08E-05	0.96

Example 2 We carry out similar numerical experiments to the variable-order tFDE (3.1) in three space dimensions. Based on Theorems 3.2-3.3, we take $u(x_1, x_2, x_3, t) = t^{2-\alpha(t)} \sin(2\pi x_1) \sin(2\pi x_2) \sin(2\pi x_3)$ and the right-hand side term f is evaluated accordingly. Other data are chosen to be the same as in Section 3.4.1. We measure the temporal convergence rate κ such that $\|u - U\|_{\hat{L}^\infty(0,T;L^2)} \leq Q\tau^\kappa$. Uniform spatial partition of $h = 1/32$ is used and we observe the first-order temporal convergence from Table 3.2, which again coincides with the conclusions in Theorem 3.6.

Table 3.2 Temporal accuracy of finite element scheme under $h = 1/32$ and different $(\alpha(0), \alpha(1))$ [112, Table 6].

τ	$(0, 0.6)$	κ	$(0.4, 0.6)$	κ	$(0, 0.4)$	κ	$(0.2, 0.4)$	κ
1/8	1.66E-02		1.53E-02		1.88E-02		1.82E-02	
1/16	8.32E-03	1.00	7.68E-03	1.00	9.42E-03	1.00	9.12E-03	1.00
1/32	4.14E-03	1.01	3.82E-03	1.01	4.70E-03	1.00	4.54E-03	1.00
1/64	2.06E-03	1.01	1.89E-03	1.02	2.33E-03	1.01	2.25E-03	1.01

Chapter 4

Variable-order time-fractional PDEs with hidden memory

In this chapter, we study the following hidden-memory variable-order tFDE model with a more general variable-index space-fractional Laplacian operator [60, 86, 87, 114, 116]

$$\begin{aligned} &\partial_t u + k\, {}_0\hat{\partial}_t^{\alpha(t)} u + \mathcal{B}^{\beta(t)} u = f(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T]; \\ &u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}),\ \boldsymbol{x} \in \Omega; \quad u(\boldsymbol{x}, t) = 0,\ (\boldsymbol{x}, t) \in \partial\Omega \times [0, T], \end{aligned} \tag{4.1}$$

whre the operator $\mathcal{B}^{\beta(t)}$ is defined by [3, 77, 91, 116]

$$\mathcal{B}^{\beta(t)} v := \sum_{i=1}^{\infty} \lambda_i^{\beta(t)} (v, \phi_i)\phi_i, \quad \forall v = \sum_{i=1}^{\infty} (v, \phi_i)\phi_i \in L^2(\Omega) \tag{4.2}$$

and the $\beta(t)$ satisfies the following conditions:

Assumption B $\beta \in C^1[0, T]$ and $0 < \beta_* \le \beta(t) \le \beta^* \le 1$.

Compared with the proofs in Chapter 3, the corresponding mathematical and numerical analysis will be significantly affected due to hidden-memory variable order. To be specific, the analysis is much more complicated as the kernel of the fractional operators can not be integrated into a closed form (cf. Step 2 in the proof of Theorem 2.5). In numerical approximations, the L-1 coefficients $b_{n,k}$ lose their monotonicity (cf. (3.25)) with respect to the index k due to the impact of the hidden memory, which played a crucial rule in the corresponding error estimates (cf. the proof of Theorem 3.6). Therefore, Novel techniques have to be developed to accommodate these issues.

Similar to Section 3.2, we apply the spectral expansion to the solution and the right-hand side term to obtain the following equivalent fODE system

$$u_i'(t) + k\,{}_0\hat{\partial}_t^{\alpha(t)} u_i(t) + \lambda_i^{\beta(t)} u_i(t) = f_i(t), \quad t \in (0,T],$$

$$u_i(0) = u_{0,i} := (u_0, \phi_i), \quad i = 1, 2, \cdots.$$

We integrate the fODEs multiplied by $\exp(\int_0^t \lambda_i^{\beta(r)} dr)$ to obtain

$$u_i(t) = -\int_0^t e^{-\int_\theta^t \lambda_i^{\beta(r)} dr} \Big[k\Big({}_0\hat{I}_\theta^{1-\alpha(\theta)} u_i'\Big) - f_i(\theta)\Big] d\theta + u_{0,i} e^{-\int_0^t \lambda_i^{\beta(r)} dr}. \tag{4.3}$$

We differentiate (4.3) to obtain an integral equation in terms of $v(t) = u_i'(t)$

$$\begin{aligned} v(t) &= -k{}_0\hat{I}_t^{1-\alpha(t)} v + k\lambda_i^{\beta(t)} \int_0^t e^{-\int_\theta^t \lambda_i^{\beta(r)} dr}\, {}_0\hat{I}_\theta^{1-\alpha(\theta)} v d\theta \\ &\quad + f_i(t) - \lambda_i^{\beta(t)} \int_0^t e^{-\int_\theta^t \lambda_i^{\beta(r)} dr} f_i(\theta) d\theta - u_{0,i}\lambda_i^{\beta(t)} e^{-\int_0^t \lambda_i^{\beta(r)} dr}. \end{aligned} \tag{4.4}$$

As the analysis of this integral equation, and thus the well-posedness of model (4.1), can be carried out in parallel with the proofs in Section 3.2, we refer these results from [114] without proof and focus our attention on the estimate of $\partial_{tt} u$.

Lemma 4.1. *If the Assumption B holds, there exists a constant $K_0 \geq 1$ that is determined by $\{\lambda_i\}_{i=1}^\infty$ but is independent of any particular λ_i such that*

$$\lambda_i^{\beta(t)-\beta(s)} e^{-\int_s^t \lambda_i^{\beta(r)} dr} \leq K_0 e^{-0.5\int_s^t \lambda_i^{\beta(r)} dr}, \quad 0 \leq s \leq t \leq T.$$

Proof. As $\lim_{t\to\infty} \|\beta\|_{C^1[0,T]} \ln t/(0.5t^{\beta_*}) = 0$, there exists a constant $K_1 \geq 1$ such that $0.5t^{\beta_*} \geq \|\beta\|_{C^1[0,T]} \ln t$ on $[K_1, \infty)$. Moreover, since $\{\lambda_i\}_{i=1}^\infty$ increases monotonically to infinity, there exists a positive integer I such that $\lambda_i \geq K_1$ for $i > I$ and $\lambda_i < K_1$ for $i \leq I$. Thus, for $i > I$ we have

$$\lambda_i^{\beta(t)-\beta(s)} e^{-\int_s^t \lambda_i^{\beta(r)} dr} = e^{-\int_s^t (\lambda_i^{\beta(r)} - \beta'(r)\ln\lambda_i) dr} \leq e^{-0.5\int_s^t \lambda_i^{\beta(r)} dr}.$$

For $i \leq I$ we simply have

$$\lambda_i^{\beta(t)-\beta(s)} e^{-\int_s^t \lambda_i^{\beta(r)} dr} \leq K_0 e^{-0.5\int_s^t \lambda_i^{\beta(r)} dr}, \quad K_0 := \max\Big\{1, \max_{1\leq i\leq I} \sup_{0\leq s\leq t\leq T} \lambda_i^{\beta(t)-\beta(s)}\Big\}.$$

We combine the proceeding estimates to finish the proof. □

Lemma 4.2. *If Assumptions A-B hold and $f_i \in C[0,T]$, equation (4.4) has a unique solution $v \in C[0,T]$ and*

$$\|v\|_{C[0,T]} \le Q_0 M_0, \quad M_0 := \lambda_i^{\beta(0)}|u_{0,i}| + \|f_i\|_{C[0,T]},\ Q_0 = Q_0\Big(\alpha^*, \beta_*, \beta^*, k, T\Big).$$

Theorem 4.3. *If Assumptions A-B hold, $u_0 \in \check{H}^{\gamma+2\beta(0)}$ and $f \in H^\kappa(\check{H}^\gamma)$ with $\gamma > d/2$ and $\kappa > 1/2$, then problem (4.1) has a unique solution $u \in C^1([0,T];\check{H}^\gamma)$ and*

$$\|u\|_{C([0,T];\check{H}^s)} \le Q_1\Big(\|u_0\|_{\check{H}^{2(\beta(0)-\beta_*)+s}} + \|f\|_{H^\kappa(0,T;\check{H}^{\max\{s-2\beta_*,0\}})}\Big),$$

$$\|u\|_{C^1([0,T];\check{H}^s)} \le Q_1\Big(\|u_0\|_{\check{H}^{2\beta(0)+s}} + \|f\|_{H^\kappa(0,T;\check{H}^s)}\Big), \quad 0 \le s \le \gamma.$$

Here $Q_1 = Q_1(\alpha^*, \beta_*, \beta^*, k, T, \kappa)$.

Theorem 4.4. *[114] Suppose Assumptions A-B hold, $u_0 \in \check{H}^{\gamma+4\beta^*}$ and $f \in H^\kappa(\check{H}^{\gamma+2\beta^*}) \cap H^{1+\kappa}(\check{H}^\gamma)$ for some $\gamma > \max\{0, d/2 - 2\beta^*\}$ and $\kappa > 1/2$. If $\alpha(0) > 0$, then $\partial_{tt} u \in C((0,T];\check{H}^s)$ for $0 \le s \le \gamma$ and*

$$\|\partial_{tt} u\|_{\check{H}^s} \le Q_2 t^{-\alpha(0)}\Big(\|u_0\|_{\check{H}^{s+4\beta^*}} + \|f\|_{H^\kappa(\check{H}^{s+2\beta^*})} + \|f\|_{H^{1+\kappa}(\check{H}^s)}\Big), \quad 0 < t \le T.$$

If $\alpha(0) = 0$, then $\partial_{tt} u \in C([0,T];\check{H}^s)$ with the global estimate

$$\|\partial_{tt} u\|_{\check{H}^s} \le Q_2\Big(\|u_0\|_{H^{s+4\beta^*}} + \|f\|_{H^\kappa(\check{H}^{s+2\beta^*})} + \|f\|_{H^{1+\kappa}(\check{H}^s)}\Big), \quad 0 \le t \le T.$$

Here $Q_2 = Q_2\Big(\alpha^*, \|\alpha\|_{C^1[0,T]}, \beta_*, \beta^*, k, T, \kappa\Big)$.

Proof. We begin with $\alpha(0) > 0$. We multiply equation (4.4) by t and use $t = s + (t-s)$ to split ${}_0\hat{I}_t^{1-\alpha(t)}(tv(s))$ to get the following equation in terms of $sv(s)$

$$\begin{aligned} tv = & -k_0 \hat{I}_t^{1-\alpha(t)}(sv(s)) - k\int_0^t \frac{(t-s)^{1-\alpha(s)} v(s)\,ds}{\Gamma(1-\alpha(s))} \\ & + t\int_0^t \lambda_i^{\beta(t)} e^{-\int_\theta^t \lambda_i^{\beta(r)}dr} k_0 \hat{I}_\theta^{1-\alpha(\theta)} v d\theta + t f_i(t) \\ & - t\int_0^t \lambda_i^{\beta(t)} e^{-\int_\theta^t \lambda_i^{\beta(r)}dr} f_i(\theta) d\theta - t\, u_{0,i} \lambda_i^{\beta(t)} e^{-\int_0^t \lambda_i^{\beta(r)}dr}. \end{aligned}$$

Since $v \in C[0,T]$ by Lemma 4.2, all but the first terms on the right-hand side are in $C^1[0,T]$. Then we apply the bootstrapping procedure like that below (3.17) to conclude that v is differentiable for $t \in (0,T]$. Consequently, u_i has a second-order derivative for $t \in (0,T]$.

To derive a stability estimate for v' on $(0,T]$, we differentiate (4.4) to obtain

$$v'(t) = -k\left({}_0\hat{I}_t^{1-\alpha(t)}v\right)_t + R, \tag{4.5}$$

Here R denotes the derivative of all but the first terms on the right-hand side of (4.4)

$$R := k\lambda_i^{\beta(t)}{}_0\hat{I}_t^{1-\alpha(t)}v - \int_0^t k\frac{\lambda_i^{2\beta(t)} - \lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}{}_0\hat{I}_\theta^{1-\alpha(\theta)}vd\theta + f_i'(t)$$
$$-\lambda_i^{\beta(t)}f_i(t) + \int_0^t \frac{\lambda_i^{2\beta(t)} - \lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}f_i(\theta)d\theta - \frac{\lambda_i^{2\beta(t)} - \lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_0^t \lambda_i^{\beta(r)}dr}}u_{0,i}.$$

To differentiate the weakly singular integral ${}_0\hat{I}_t^{1-\alpha(t)}v$, we first integrate the v in ${}_0\hat{I}_t^{1-\alpha(t)}$ by parts. However, the kernel $(t-s)^{-\alpha(s)}$ cannot be integrated in a closed form. Instead, we integrate its leading part $(t-s)^{-\alpha(t)}$ by parts to obtain

$$\begin{aligned}
{}_0\hat{I}_t^{1-\alpha(t)}v &= -\frac{1}{1-\alpha(t)}\int_0^t \frac{v(s)d(t-s)^{1-\alpha(t)}}{\Gamma(1-\alpha(s))(t-s)^{\alpha(s)-\alpha(t)}} =: \frac{\eta_v(t)}{1-\alpha(t)},\\
\eta_v(t) &:= \frac{v(0)t^{1-\alpha(0)}}{\Gamma(1-\alpha(0))} + \int_0^t (t-s)^{1-\alpha(s)}\left[\frac{\Gamma'(1-\alpha(s))\alpha'(s)v(s)}{\Gamma(1-\alpha(s))^2}\right.\\
&\quad \left. + \frac{v'(s)}{\Gamma(1-\alpha(s))} - \frac{v(s)}{\Gamma(1-\alpha(s))}\left(\alpha'(s)\ln(t-s) + \frac{\alpha(t)-\alpha(s)}{t-s}\right)\right]ds.
\end{aligned}$$

We consequently get

$$\begin{aligned}
\left({}_0\hat{I}_t^{1-\alpha(t)}v\right)' &= \frac{\eta_v'(t)}{1-\alpha(t)} + \frac{\alpha'(t)\eta_v(t)}{(1-\alpha(t))^2},\\
\eta_v'(t) &= \frac{v(0)t^{-\alpha(0)}}{\Gamma(-\alpha(0))} + {}_0\hat{I}_t^{1-\alpha(t)}\Big((1-\alpha(s))v'(s) - (1-\alpha(s))\alpha'(s)v(s)\Big[\ln(t-s)\\
&\quad - \frac{\Gamma'(1-\alpha(s))}{\Gamma(1-\alpha(s))}\Big] - v(s)\Big[\alpha'(s) + \alpha'(t) - \frac{\alpha(s)(\alpha(t)-\alpha(s))}{t-s}\Big]\Big).
\end{aligned}$$

We incorporate this with

$$\begin{aligned}
\frac{|\ln(t-s)|}{(t-s)^{\alpha(s)}} &= (t-s)^{\alpha^*-\alpha(s)}\frac{|\ln(t-s)|}{(t-s)^{\alpha^*}} \le \max\{1,T\}\frac{|\ln(t-s)|}{(t-s)^{\alpha^*}}\\
&= \max\{1,T\}\frac{|\ln(t-s)|(t-s)^{(1-\alpha_*)/2}}{(t-s)^{(1+\alpha^*)/2}} \le \frac{Q}{(t-s)^{(1+\alpha^*)/2}}
\end{aligned}$$

and

$$
\begin{aligned}
{}_0\hat{I}_t^{1-\alpha(t)}|v(s)\ln(t-s)| &= \int_0^t \frac{|v(s)\ln(t-s)|}{\Gamma(1-\alpha(s))(t-s)^{\alpha(s)}}ds \\
&= \int_0^t \frac{|\ln(t-s)|}{\Gamma(1-\alpha(s))(t-s)^{\alpha(s)}}\Big|\int_0^s v'(y)dy + v(0)\Big|ds \\
&\le Q\Big(\int_0^t |v'(y)|dy + |v(0)|\Big)\int_0^t \frac{1}{(t-s)^{(1+\alpha^*)/2}}ds \le Q\Big(\int_0^t |v'(y)|dy + |v(0)|\Big),
\end{aligned}
$$

as well as Lemma 4.2 and similar techniques in (3.7) to bound the second term on the right-hand side of (4.5) by

$$
\Big|k\big({}_0\hat{I}_t^{1-\alpha(t)}v\big)'\Big| \le Q\int_0^t \frac{|v'(s)|ds}{(t-s)^{\alpha^*}}ds + \big(\lambda_i^{\beta(0)}|u_{0,i}| + \|f_i\|_{C[0,T]}\big)t^{-\alpha(0)}.
$$

We apply Lemma 4.2, Lemma 2.5 and the following estimate based on Lemma 4.1

$$
\begin{aligned}
\lambda^{\beta(t)}\int_0^t e^{-\int_\theta^t \lambda^{\beta(r)}dr}d\theta &= \int_0^t \frac{\lambda^{\beta(t)-\beta(\theta)}\lambda^{\beta(\theta)}}{e^{\int_\theta^t \lambda^{\beta(r)}dr}}d\theta \le \int_0^t \frac{K_0\lambda^{\beta(\theta)}d\theta}{e^{0.5\int_\theta^t \lambda^{\beta(r)}dr}} \\
&= 2K_0 e^{-0.5\int_\theta^t \lambda^{\beta(r)}dr}\Big|_{\theta=0}^{\theta=t} = 2K_0\Big(1-e^{-0.5\int_0^t \lambda^{\beta(r)}dr}\Big) \le 2K_0
\end{aligned}
$$

to bound R in (4.5) by

$$
\begin{aligned}
&\Big|k(t)\lambda_i^{\beta(t)}{}_0\hat{I}_t^{1-\alpha(t)}v - \int_0^t k(\theta)\frac{\lambda_i^{2\beta(t)}-\lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}{}_0\hat{I}_\theta^{1-\alpha(\theta)}vd\theta\Big| \\
&\quad\le Q\|v\|_{C[0,T]}\Big(\lambda_i^{\beta(t)}{}_0\hat{I}_t^{1-\alpha(t)}1 + \int_0^t \frac{\lambda_i^{2\beta(t)}}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}{}_0\hat{I}_\theta^{1-\alpha(\theta)}1d\theta\Big) \\
&\quad\le Q\|v\|_{C[0,T]}\lambda_i^{\beta^*}\Big(1+\int_0^t \frac{\lambda_i^{\beta(t)}d\theta}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}\Big) \le Q\lambda_i^{\beta^*}\big(\lambda_i^{\beta(0)}|u_{0,i}| + \|f_i\|_{C[0,T]}\big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|f_i' - \lambda_i^{\beta(t)}f_i(t) + \int_0^t \frac{\lambda_i^{2\beta(t)}-\lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_\theta^t \lambda_i^{\beta(r)}dr}}f_i(\theta)d\theta - \frac{\lambda_i^{2\beta(t)}-\lambda_i^{\beta(t)}\beta'(t)\ln\lambda_i}{e^{\int_0^t \lambda_i^{\beta(r)}dr}}u_{0,i}\Big| \\
&\quad\le Q\big(\lambda_i^{2\beta^*}|u_{0,i}| + \lambda_i^{\beta^*}\|f_i\|_{C[0,T]} + \|f_i\|_{C^1[0,T]}\big) =: M_1.
\end{aligned}
$$

We incorporate the preceding estimates into (4.5) to get

$$
|v'(t)| \quad \le Q\int_0^t \frac{|v'(s)|ds}{(t-s)^{\alpha^*}} + QM_1 t^{-\alpha(0)}, \quad t\in(0,T].
$$

The rest of the proof can be performed in parallel with those under (3.19) and is thus omitted. □

We follow [114] to present and analyze a spectral Galerkin approximation to model (4.1). Define a uniform temporal partition on $[0,T]$ by $t_n := n\tau$ for $\tau := T/N$ and $0 \le n \le N$. Let $S_M := \mathrm{span}\{\phi_i(\boldsymbol{x})\}_{i=1}^M$ with $\{\phi_i\}_{i=1}^\infty$ being introduced in (2.1). Let $u_n := u(\boldsymbol{x}, t_n)$, we discretize $\partial_t u$ and ${}_0\hat{\partial}_t^{\alpha(t)} u$ at $t = t_n$ for $1 \le n \le N$ by

$$\begin{aligned}
&\partial_t u(\boldsymbol{x}, t_n) = \delta_\tau u_n + E_n := \frac{u_n - u_{n-1}}{\tau} + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \partial_{tt} u(\boldsymbol{x}, t)(t - t_{n-1})dt,\\
&{}_0\hat{\partial}_t^{\alpha(t_n)} u(\boldsymbol{x}, t_n) = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\alpha(s))(t_n - s)^{\alpha(s)}} ds = \delta_\tau^{\alpha(t_n)} u_n + \hat{R}_n + R_n\\
&= \sum_{k=1}^n \bigg[\int_{t_{k-1}}^{t_k} \frac{\delta_\tau u_k}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds + \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\alpha(s))(t_n - s)^{\alpha(s)}} ds\\
&\quad - \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds + \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s) - \delta_\tau u_k}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds \bigg].
\end{aligned} \tag{4.6}$$

Here $\delta_\tau^{\alpha(t_n)} u_n$, $\hat{R}_n$ and R_n are defined by

$$\begin{aligned}
&\delta_\tau^{\alpha(t_n)} u_n := \sum_{k=1}^n b_{n,k}(u_k - u_{k-1}),\\
&b_{n,k} := \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \frac{1}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds = \frac{(t_n - t_{k-1})^{1-\alpha(t_k)} - (t_n - t_k)^{1-\alpha(t_k)}}{\Gamma(2-\alpha(t_k))\tau},\\
&\hat{R}_n := \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \bigg[\frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\alpha(s))(t_n - s)^{\alpha(s)}} - \frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} \bigg] ds,\\
&R_n := \sum_{k=1}^n R_{n,k} := \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s) - \delta_\tau u_k}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds\\
&\quad = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \frac{1}{\tau\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} \bigg[\int_{t_{k-1}}^{t_k} \int_z^s \partial_{\theta\theta} u(\boldsymbol{x}, \theta) d\theta dz \bigg] ds.
\end{aligned}$$

We plug (4.6) into (4.1), and integrate the resulting equation multiplied by $\chi \in \check{H}^{\beta^*}(\Omega)$ on Ω to obtain the following equation for $n = 1, 2, \dots, N$

$$\begin{aligned}
&(\delta_\tau u_n, \chi) + k(\delta_\tau^{\alpha(t_n)} u_n, \chi) + \Big(\mathcal{A}^{\beta(t_n)/2} u_n, \mathcal{A}^{\beta(t_n)/2}\chi\Big)\\
&\qquad = (f(\cdot, t_n), \chi) - \Big(k(t_n)(\hat{R}_n + R_n) + E_n, \chi\Big), \quad \chi \in \check{H}^{\beta^*}(\Omega).
\end{aligned} \tag{4.7}$$

We drop the last right-hand side term to obtain a spectral Galerkin scheme for problem (4.1): find $U_n \in S_M$ for $n = 1, 2, \dots, N$, with $U_0(\boldsymbol{x}) := \Pi_M u_0(\boldsymbol{x})$, such that

$$(\delta_\tau U_n, \chi) + k(\delta_\tau^{\alpha(t_n)} U_n, \chi) + \Big(\mathcal{A}^{\beta(t_n)/2} U_n, \mathcal{A}^{\beta(t_n)/2}\chi\Big) = (f(\cdot, t_n), \chi), \quad \chi \in S_M. \tag{4.8}$$

Here $\Pi_M : L^2 \to S_M$ is defined by $\Pi_M g := \sum_{i=1}^{M}(g,\phi_i)\phi_i, \forall g = \sum_{i=1}^{\infty}(g,\phi_i)\phi_i \in L^2$.

By (4.2) and (2.2), for any $g \in \check{H}^\gamma$ with $\gamma \geq 0$

$$\left\|g - \Pi_M g\right\|^2 = \sum_{i=M+1}^{\infty} (g,\phi_i)^2 = \sum_{i=M+1}^{\infty} \lambda_i^{-\gamma}\lambda_i^{\gamma}(g,\phi_i)^2 \leq \lambda_{M+1}^{-\gamma}\|g\|_{\check{H}^\gamma}^2,$$

which yields [8, 82]

$$\left\|g - \Pi_M g\right\| \leq \lambda_{M+1}^{-\gamma/2}\|g\|_{\check{H}^\gamma}, \quad \forall g \in \check{H}^\gamma. \tag{4.9}$$

The estimates of the truncation errors can be carried out by similar techniques as those in 3.3.1 and thus we omit the proof. Detailed derivations can be found in [114].

Lemma 4.5. *Suppose Assumptions A-B hold,* $u_0 \in \check{H}^{4\beta^*}$ *and* $f \in H^\kappa(\check{H}^{2\beta^*}) \cap H^{1+\kappa}(L^2)$ *for* $\kappa > 1/2$. *Then the time-step wise estimates hold*

$$\|E_n\| \leq Q\hat{Q}_0(N/n)^{\alpha(0)}\tau, \quad \|\hat{R}_n\| \leq Q\hat{Q}_0\tau, \quad \|R_n\| \leq Q\hat{Q}_0(N/n)^{\alpha^*}\tau$$

for $1 \leq n \leq N$ *with* $\hat{Q}_0 := \|u_0\|_{\check{H}^{4\beta^*}} + \|f\|_{H^\kappa(\check{H}^{2\beta^*})} + \|f\|_{H^{1+\kappa}(L^2)}$.

If Assumptions A-B hold, $\alpha \in C^1[0,T]$, $u_0 \in \check{H}^{2\beta(0)+s}$ *and* $f \in H^\kappa(\check{H}^s)$ *for* $\kappa > 1/2$ *and* $s \geq 0$, *then* $\eta_n := \Big(I - \Pi_M\Big)u(\boldsymbol{x}, t_n)$ *are bounded by*

$$\|\delta_\tau \eta_n\|_{\hat{L}^\infty(0,T;L^2)} := \max_{1\leq n\leq N}\left\|\delta_\tau \eta_n\right\| \leq Q\hat{Q}_1\lambda_{M+1}^{-s/2}, \quad \|\delta_\tau^{\alpha(t_n)}\eta_n\|_{\hat{L}^\infty(0,T;L^2)} \leq Q\hat{Q}_1\lambda_{M+1}^{-s/2}.$$

Here I *is the identity operator and* $\hat{Q}_1 := \|u_0\|_{\check{H}^{2\beta(0)+s}} + \|f\|_{H^\kappa(\check{H}^s)}$.

4.2.1 Optimal-order error estimate of spectral Galerkin scheme (4.8)

We note from the expression of $b_{n,k}$ below (4.6) that the power and the denominator all depend on t_k, due to the hidden memory impact of the $\alpha(s)$ in problem (4.1). Consequently, $b_{n,k}$ lose their monotonicity with respect to the index k. Recall that in the context of constant-order tFDE (3.2), α is constant. Hence, the corresponding coefficient $b_{n,k}$ is monotonically decreasing with respect to k, which played a crucial rule in the corresponding error analysis [85, 89]. In the context of variable-order tFDE

(3.1), the power and the denominator of $b_{n,k}$ depended only on n. Hence, at any time step t_n, $b_{n,k}$ still enjoyed the monotonicity. Its clever application contributes to the error analysis of an L-1 temporal discretization of problem (3.1) [112].

To overcome the difficulty that $b_{n,k}$ lose their monotonicity in the current context, we decompose each $b_{n,k} - b_{n,k-1}$ as the sum of a positive-preserving term and a high-order perturbation. To do so, we introduce an auxiliary sequence $\{\hat{b}_{n,k}\}_{k=1}^{n-1}$ defined by

$$\hat{b}_{n,k} := \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \frac{1}{\Gamma(1-\alpha(t_{k+1}))(t_n - s)^{\alpha(t_{k+1})}} ds.$$

Lemma 4.6. *There is a positive constant $\hat{Q}_2 > 0$, independent of n, N, τ such that*

$$b_{n,k+1} > \hat{b}_{n,k}, \quad 1 \le k \le n-1, \quad \sum_{k=1}^{n-1} |\hat{b}_{n,k} - b_{n,k}| \le \hat{Q}_2, \quad 1 \le n \le N. \tag{4.10}$$

Consequently, the following holds for any non-negative sequence $\{z_k\}_{k=1}^N$

$$\sum_{k=1}^{n-1} \Big(b_{n,k+1} - b_{n,k}\Big) z_k \le \sum_{k=1}^{n-1} \Big(b_{n,k+1} - \hat{b}_{n,k} + |\hat{b}_{n,k} - b_{n,k}|\Big)|z_k|, \quad 1 \le n \le N. \tag{4.11}$$

Proof. By the definitions of $b_{n,k}$ below (4.6) and $\hat{b}_{n,k}$ we have

$$\begin{aligned}
&b_{n,k+1} - \hat{b}_{n,k} \\
&= \frac{1}{\tau}\bigg(\int_{t_k}^{t_{k+1}} \frac{ds}{\Gamma(1-\alpha(t_{k+1}))(t_n - s)^{\alpha(t_{k+1})}} - \int_{t_{k-1}}^{t_k} \frac{ds}{\Gamma(1-\alpha(t_{k+1}))(t_n - s)^{\alpha(t_{k+1})}}\bigg) \\
&= \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \frac{1}{\Gamma(1-\alpha(t_{k+1}))}\bigg(\frac{1}{(t_n - s - \tau)^{\alpha(t_{k+1})}} - \frac{1}{(t_n - s)^{\alpha(t_{k+1})}}\bigg) ds > 0.
\end{aligned}$$

We use the mean-value theorem and

$$\begin{aligned}
\frac{|\ln(t_n - s)|}{(t_n - s)^{\alpha(\zeta)}} &= (t-s)^{\alpha^* - \alpha(\zeta)} \frac{|\ln(t-s)|}{(t-s)^{\alpha^*}} \le \max\{1, T\} \frac{|\ln(t-s)|}{(t-s)^{\alpha^*}} \\
&= \max\{1, T\} \frac{|\ln(t-s)|(t-s)^{(1-\alpha_*)/2}}{(t-s)^{(1+\alpha^*)/2}} \le \frac{Q}{(t-s)^{(1+\alpha^*)/2}}
\end{aligned}$$

to get

$$\begin{aligned}
|\hat{b}_{n,k} - b_{n,k}| &= \frac{1}{\tau}\bigg|\int_{t_{k-1}}^{t_k} \frac{1}{\Gamma(1-\alpha(t_{k+1}))(t_n - s)^{\alpha(t_{k+1})}} - \frac{1}{\Gamma(1-\alpha(t_k))(t_n - s)^{\alpha(t_k)}} ds\bigg| \\
&= \bigg|\int_{t_{k-1}}^{t_k} \frac{\Gamma'(1-\alpha(\zeta))\alpha'(\zeta)}{\Gamma^2(1-\alpha(\zeta))(t_n - s)^{\alpha(\zeta)}} - \frac{\ln(t_n - s)\alpha'(\zeta)}{\Gamma(1-\alpha(\zeta))}(t_n - s)^{\alpha(\zeta)} ds\bigg| \\
&\le Q\int_{t_{k-1}}^{t_k} \frac{1}{(t_n - s)^{(1+\alpha^*)/2}} ds = \frac{2Q\Big[\big(t_n - t_{k-1}\big)^{\frac{1-\alpha^*}{2}} - \big(t_n - t_k\big)^{\frac{1-\alpha^*}{2}}\Big]}{1-\alpha^*},
\end{aligned}$$

$$\sum_{k=1}^{n-1} |\hat{b}_{n,k} - b_{n,k}| \le Q \sum_{k=1}^{n-1} \Big[\Big(t_n - t_{k-1}\Big)^{\frac{1-\alpha^*}{2}} - \Big(t_n - t_k\Big)^{\frac{1-\alpha^*}{2}} \Big] \le Q.$$

The inequality (4.11) follows from $b_{n,k+1} > \hat{b}_{n,k}$ in (4.10). □

Finally, we are in the position to prove the main theorem of this paper.

Theorem 4.7. *Suppose Assumptions A-B hold,* $u_0 \in \check{H}^{\max\{4\beta^*, 2\beta(0)+s\}}$ *and* $f \in H^\kappa(\check{H}^{\max\{2\beta^*, s\}}) \cap H^{1+\kappa}(L^2)$ *for* $\kappa > 1/2$ *and* $s \ge 0$*. Then an optimal-order error estimate holds for the spectral Galerkin scheme* (4.8)

$$\|U - u\|_{\hat{L}^\infty(0,T;L^2)} \le Q\Big(\hat{Q}_0 \tau + \hat{Q}_1 \lambda_{M+1}^{-s/2}\Big).$$

Here the positive constant $Q = Q(\alpha^*, T, \|\alpha\|_{C^1[0,T]}, k)$*, and* $\hat{Q}_0$ *and* $\hat{Q}_1$ *are introduced in Lemma 4.5.*

Proof. We split $u_n - U_n = \xi_n + \eta_n$ where $\xi_n := \Pi_M u_n - U_n$, with Π_M defined below (4.8), and η_n was bounded in (4.9). Hence, we remain to bound ξ_n. We subtract (4.8) from (4.7) with $\chi = \xi_n$ to obtain the following error equation in terms of ξ_n

$$\begin{aligned} &(\delta_\tau \xi_n, \xi_n) + \Big(\mathcal{A}^{\beta(t)/2}\xi_n, \mathcal{A}^{\beta(t)/2}\xi_n\Big) + k(\delta_\tau^{\alpha(t_n)}\xi_n, \xi_n) \\ &\quad = \Big(k[\hat{R}_n + R_n - \delta_\tau^{\alpha(t_n)}\eta_n] + E_n - \delta_{\tau_n}\eta_n, \xi_n\Big). \end{aligned} \tag{4.12}$$

We use $\xi_0 := U_0 - \Pi_M u_0 = 0$ to rewrite $\delta_\tau^{\alpha(t_n)}\xi_n = b_{n,n}\xi_n - \sum_{k=1}^{n-1}\Big(b_{n,k+1} - b_{n,k}\Big)\xi_k$ and reformulate (4.12) as

$$\begin{aligned} &\Big[1 + \tau k b_{n,n}\Big]\|\xi_n\|^2 + \tau\Big\|\mathcal{A}^{\beta(t)/2}\xi_n\|^2 \\ &\quad = (\xi_{n-1}, \xi_n) + \tau k \sum_{k=1}^{n-1}\Big(b_{n,k+1} - b_{n,k}\Big)(\xi_k, \xi_n) \\ &\qquad + \tau\Big(k\Big[\hat{R}_n + R_n - \delta_\tau^{\alpha(t_n)}\eta_n\Big] + E_n - \delta_\tau\eta_n, \xi_n\Big). \end{aligned}$$

We use Cauchy inequality to cancel $\|\xi_n\|$ on both sides and use (4.11) to obtain

$$\begin{aligned} &\Big(1 + \tau k b_{n,n}\Big)\|\xi_n\| \\ &\quad \le \|\xi_{n-1}\| + \tau k \sum_{k=1}^{n-1}\Big(b_{n,k+1} - \hat{b}_{n,k} + |\hat{b}_{n,k} - b_{n,k}|\Big)\|\xi_k\| + \tau G_n, \end{aligned} \tag{4.13}$$

where G_n is defined by

$$G_n := k\Big(\big\|\hat{R}_n\big\| + \big\|R_n\big\| + \big\|\delta_\tau^{\alpha(t_n)}\eta_n\big\|\Big) + \big\|E_n\big\| + \big\|\delta_\tau\eta_n\big\|.$$

By (4.13) $\|\xi_1\| \le \tau G_1 \le \tau(1+2\hat{Q}_2 k\tau)G_1$ with $\hat{Q}_2$ given in (4.10). Assume

$$\|\xi_m\| \le A_m \sum_{j=1}^{m} G_j, \quad A_m := \tau\Big(1+2\hat{Q}_2 k\tau\Big)^m, \quad 1 \le m \le n-1. \tag{4.14}$$

We plug (4.14) with $2 \le m \le n-1$ into (4.13), use $A_N > A_{N-1} > \cdots > A_1 > \tau$ and

$$\begin{aligned}
&\sum_{k=1}^{n-1}\Big(b_{n,k+1} - \hat{b}_{n,k} + |\hat{b}_{n,k} - b_{n,k}|\Big) \\
&\quad = \sum_{k=1}^{n-1}\Big(b_{n,k+1} - \hat{b}_{n,k} + (\hat{b}_{n,k} - b_{n,k}) - (\hat{b}_{n,k} - b_{n,k}) + |\hat{b}_{n,k} - b_{n,k}|\Big) \\
&\quad \le \sum_{k=1}^{n-1}\Big(b_{n,k+1} - b_{n,k}\Big) + 2\sum_{k=1}^{n-1}|\hat{b}_{n,k} - b_{n,k}| \le b_{n,n} + 2\hat{Q}_2 \text{ (using (4.10))}
\end{aligned}$$

to arrive at the following bound

$$\begin{aligned}
&(1+\tau k b_{n,n})\|\xi_n\| \\
&\quad \le A_{n-1}\sum_{j=1}^{n-1} G_j + \tau k\Big[A_{n-1}\sum_{j=1}^{n-1} G_j\Big]\sum_{k=1}^{n-1}\Big(b_{n,k+1} - \hat{b}_{n,k} + |\hat{b}_{n,k} - b_{n,k}|\Big) + \tau G_n \\
&\quad \le A_{n-1}\sum_{j=1}^{n} G_j + \tau k\Big[A_{n-1}\sum_{j=1}^{n} G_j\Big]\Big(b_{n,n} + 2\hat{Q}_2\Big) \\
&\quad = \Big[A_{n-1}\sum_{j=1}^{n} G_j\Big]\Big(1 + \tau k b_{n,n} + 2k(t_n)\hat{Q}_2\tau\Big).
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\|\xi_n\| &\le \Big[A_{n-1}\sum_{j=1}^{n} G_j\Big]\Big(1 + \frac{2k\hat{Q}_2\tau}{1+\tau k b_{n,n}}\Big) \\
&\le \Big[A_{n-1}\sum_{j=1}^{n} G_j\Big]\Big(1 + 2\hat{Q}_2 k\tau\Big) = A_n\sum_{j=1}^{n} G_j.
\end{aligned}$$

Thus, (4.14) holds for $m = n$ and so for any $m \ge 2$ by mathematical induction.

We remain to bound the right-hand side of (4.14) for any $1 \le m \le N$. As $(1+2Q_2k\tau)^N \le Q$, it suffices to bound $\tau\sum_{n=1}^N G_n$. We use Lemmas 3.4 and 3.5 and the fact that $\sum_{n=1}^N n^{-\gamma} \le QN^{1-\gamma}$, with $\gamma = \alpha(0)$ or α^*, to conclude that

$$\begin{aligned}
\tau\sum_{n=1}^{N} G_n &\le Q\tau\sum_{n=1}^{N}\Big(\hat{Q}_0\tau\Big[\Big(\frac{N}{n}\Big)^{\alpha(0)} + \Big(\frac{N}{n}\Big)^{\alpha^*} + 1\Big] + \hat{Q}_1\lambda_{M+1}^{-s/2}\Big) \\
&\le Q\Big(\hat{Q}_0\tau + \hat{Q}_1\lambda_{M+1}^{-s/2}\Big).
\end{aligned}$$

We combine this estimate and (4.9) to complete the proof. □

4.3 Numerical experiments

We follow [114] carry out numerical experiments to investigate the regularity of the solutions to model (4.1) and its dependence on the behavior of the variable order $\alpha(t)$ as well as the temporal convergence of the spectral Galerkin approximation to problem (4.1). In numerical experiments we assume a rectangular domain $\Omega = (0,1)^d$, and the spectral Galerkin subspace

$$S_M := \operatorname{span}\Big\{\phi_{i_1}(x_1)\cdots\phi_{i_d}(x_d)\Big\}_{i_1=1,\cdots,i_d=1}^{M,\cdots,M}. \tag{4.15}$$

Here $\phi_i(x_j) := \sqrt{2}\sin(i\pi x_j)$ is the i-th basis function in the j-th direction for $1 \le j \le d$ and the corresponding eigenvalue of $\phi^1_{i_1}(x_1) \times \cdots \times \phi_{i_d}(x_d)$ is $\pi^2(i_1^2 + \cdots + i_d^2)$.

4.3.1 Behavior of the solutions to model (4.1)

The data are as follows: $\Omega = (0,1)$, $[0,T] = [0,1]$, $k(t) = 1$, $\boldsymbol{K} = K := 0.001$, $f = 0$, $u_0(x) = x^4(1-x)^4$, and the variable orders $\alpha(t)$ and $\beta(t)$ are given by

$$\alpha(t) = \alpha(T) + (\alpha(0) - \alpha(T))\Big(1 - t/T - \sin(2\pi(1-t/T))/(2\pi)\Big),$$

$$\beta(t) = \beta(T) + (\beta(0) - \beta(T))\Big(1 - t/T - \sin(2\pi(1-t/T))/(2\pi)\Big).$$

We present first-order time difference quotients $\delta_\tau U_n(1/2)$ of the numerical solutions to problem (4.1) in Figure 4.1 with $(\beta(0), \beta(1)) = (0.8, 0.2)$, $N = 1600$ and $M = 200$ and $d = 1$ in (4.15) for the three cases:

(i) $\alpha(0) = 0,\ \alpha(1) = 0.9$; (ii) $\alpha(0) = 0.5,\ \alpha(1) = 0.9$; (iii) $\alpha(0) = 0.8,\ \alpha(1) = 0.9$.

We observe that the solution for case (i) is smooth near the initial time $t = 0$ but those for cases (ii) and (iii) exhibit initial weak singularities near the initial time $t = 0$ and the singularities get stronger as $\alpha(0)$ increases. These observations numerically justify the analysis in Theorem 4.4.

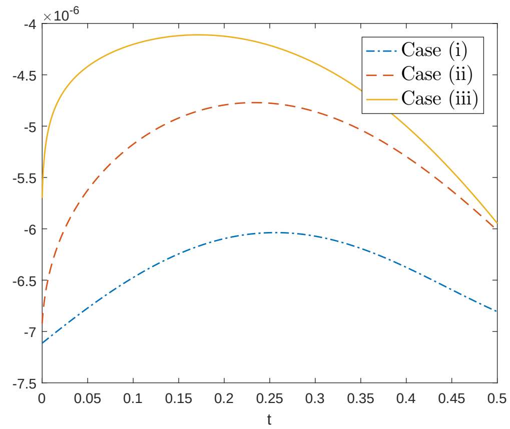

Figure 4.1 First-order time different quotients $\delta_\tau U_n(1/2, t)$ of the solutions to problem (4.1) for cases (i)–(iii) [114, FIG. 5.1].

4.3.2 Convergence of scheme (4.8)

We numerically investigate the temporal convergence behavior of the spectral-Galerkin scheme (4.8) to problem (4.1).

Example 1 We simulate the example in §4.3.1 with (a) $\alpha(0) = 0.8$, $\alpha(1) = 0.9$, $\beta(0) = 0.8$, $\beta(1) = 0.2$ and (b) $\alpha(0) = 0.1$, $\alpha(1) = 0.9$, $\beta(0) = 0.2$, and $\beta(1) = 0.5$. As closed-form solutions to model (4.1) are not available, we use a numerical solution $\hat{U}$ with $M = 200$ and $N = 1600$ as the reference solution to test the temporal convergence of the scheme by $\|U - \hat{U}\|_{\hat{L}^\infty(0,T;L^2)} \le Q\tau^\kappa$. We present the numerical results in Table 4.1, which show the first-order convergence in time of scheme (4.8) as proved in Theorem 4.7.

Table 4.1 Temporal convergence of scheme (4.8) in Example 1 with $M = 200$ [114, Table 5.1].

τ	(a)	κ	(b)	κ
1/30	1.80E-08		2.28E-07	
1/40	1.31E-08	1.11	1.70E-07	1.02
1/50	1.02E-08	1.11	1.35E-07	1.03
1/60	8.34E-09	1.11	1.12E-07	1.03

Example 2 The data are $\Omega = (0,1)^3$, $[0,T] = [0,1]$, $k(t) = 1$, $\boldsymbol{K} = \text{diag}(0.001, 0.001, 0.001)$, $f = 0$,

$$u_0(x_1, x_2, x_3) = e^{-[(x_1-1/2)^2+(x_2-1/2)^2+(x_3-1/2)^2]/0.01},$$

and the variable orders $\alpha(t)$ and $\beta(t)$ are given by

$$\alpha(t) = \alpha(T) + (\alpha(0) - \alpha(T))\Big(1 - t/T\Big), \quad \beta(t) = \beta(T) + (\beta(0) - \beta(T))\Big(1 - t/T\Big).$$

We investigate convergence rates for (c) $\alpha(0) = 0.3$, $\alpha(1) = 0.6$, $\beta(0) = 0.1$, $\beta(1) = 0.1$ and (d) $\alpha(0) = 0.9$, $\alpha(1) = 0.3$, $\beta(0) = 0.02$, and $\beta(1) = 0.12$. We use a numerical solution with $M = 30$ and $N = 1200$ as the reference solution to test the temporal convergence of the scheme and present the numerical results in Table 4.2, which again show the first-order convergence in time of scheme (4.8) as proved in Theorem 4.7.

Table 4.2 Temporal convergence of scheme (4.8) in Example 2 with $M = 30$ [114, Table 5.2].

τ	(c)	κ	(d)	κ
1/50	2.40E-07		6.87E-08	
1/60	2.00E-07	1.02	5.67E-08	1.06
1/70	1.70E-07	1.03	4.81E-08	1.06
1/80	1.48E-07	1.04	4.17E-08	1.07

Chapter 5

Time-fractional PDEs with space-time dependent variable order

In this chapter, we follow [124] to investigate a more general time-fractional diffusion equation with a hidden-memory space-time dependent variable order

$$\begin{aligned} &\partial_t u(\boldsymbol{x},t) + k\partial_t^{\gamma(\boldsymbol{x},t)} u(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T], \\ &u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \partial\Omega \times (0,T], \qquad u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \end{aligned} \tag{5.1}$$

where the hidden-memory space-time dependent variable order fractional derivative operator $\partial_t^{\gamma(\boldsymbol{x},t)}$ of order $0 \le \gamma(\boldsymbol{x},t) \le \gamma^* < 1$ is defined by [60, 41, 86, 105, 126]

$$\partial_t^{\gamma(\boldsymbol{x},t)} g := {}_0I_t^{1-\gamma(\boldsymbol{x},t)} \partial_t g, \quad {}_0I_t^{1-\gamma(\boldsymbol{x},t)} g := \int_0^t \frac{g(\boldsymbol{x},s)ds}{\Gamma(1-\gamma(\boldsymbol{x},s))(t-s)^{\gamma(\boldsymbol{x},s)}}. \tag{5.2}$$

It is clear that the operators in (5.2) generalizes the definitions (2.6) of the hidden-memory variable-order operators by imposing the space-dependence in the variable order. Therefore, the variable separation method used in previous chapters does not apply. Consequently, we alternatively employ the Laplace transform method and resolvent estimates to circumvent this issue and provide rigorous mathematical and numerical analysis to this problem.

5.1 Solution representation and resolvent estimates

For $\theta \in (\pi/2, \pi)$ and $\delta > 0$, let Γ_θ be the contour in the complex plane defined by

$$\Gamma_\theta := \Big\{ z \in \mathbb{C} : |\arg(z)| = \theta, |z| \ge \delta \Big\} \cup \Big\{ z \in \mathbb{C} : |\arg(z)| \le \theta, |z| = \delta \Big\}.$$

The following inequalities hold for $1 \le p \le \infty$, $0 < \mu \le 1$ and $t \in (0, T]$ [38, 61, 62]

$$\int_{\Gamma_\theta} |z|^{\mu-1}|e^{tz}|\,|dz| \le Qt^{-\mu}, \qquad \left\| \int_{\Gamma_\theta} z^{\mu}(z-\Delta)^{-1}e^{tz}\,dz \right\|_{L^p} \le Qt^{-\mu} \tag{5.3}$$

where $|dz|$ denotes the arc length element on the contour Γ_θ and $Q = Q(\theta, \mu, p)$.

For any $q \in L_{loc}(0, T)$, the Laplace transform $\mathcal{L}$ of its extension $\tilde{q}(t)$ with compact support on $(0, T)$ and the corresponding inverse transform $\mathcal{L}^{-1}$ are denoted by

$$\mathcal{L}q(z) := \int_0^\infty q(t)e^{-tz}dt, \qquad \mathcal{L}^{-1}(\mathcal{L}q(z)) := \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{tz}\mathcal{L}q(z)dz = q(t). \tag{5.4}$$

$\mathcal{L}q$ is always interpreted as the Laplace transform of $\tilde{q}$. It is known that [75]

$$\mathcal{L}\Big({}_0^R\partial_t^\gamma q(t)\Big) = z^\gamma \mathcal{L}\Big(q(t)\Big), \quad 0 \le \gamma < 1. \tag{5.5}$$

The solutions $u(\boldsymbol{x}, t)$ to the heat equation

$$\begin{aligned} \partial_t u(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, T], \quad u(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \Omega \end{aligned} \tag{5.6}$$

can be expressed as

$$u(\boldsymbol{x}, t) = \int_0^t E(t-s)f(\boldsymbol{x}, s)ds, \tag{5.7}$$

where $E(t) = e^{t\Delta}$, with $t \ge 0$, is the semigroup of operators generated by the Dirichlet Laplacian. Namely, $E(t)\psi$ represents the solution to the problem

$$\begin{aligned} \partial_t E(t)\psi = \Delta E(t)\psi, \\ E(t)\psi = 0, \ \boldsymbol{x} \in \partial\Omega, \quad E(0)\psi = \psi, \ \boldsymbol{x} \in \Omega \end{aligned}$$

which is given by

$$E(t)\psi(\boldsymbol{x}) := \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{zt}(z-\Delta)^{-1}\psi(\boldsymbol{x})\,dz, \qquad \forall \psi \in L^2(\Omega). \tag{5.8}$$

Moreover, $E(t) = e^{t\Delta}$ has the spectral decomposition in terms of $\{\phi_i\}_{i=1}^\infty$ in (2.1)

$$E(t)\psi(\boldsymbol{x}) = \sum_{i=1}^\infty e^{-\lambda_i t}(\psi, \phi_i)\phi_i(\boldsymbol{x}).$$

The following estimates hold for any $t > 0$ and $1 \le p \le \infty$ [91]

$$\|E(t)\|_{L^p \to L^p} + t\|E'(t)\|_{L^p \to L^p} + t\|\Delta E(t)\|_{L^p \to L^p} \le Q,$$

$$\|E(t)\psi\|_{\check{H}^s} \le Qt^{-(s-r)/2}\|\psi\|_{\check{H}^r}, \qquad \psi \in \check{H}^r, \quad s \ge r \ge -1.$$

We shall also use the well-posedness result of the heat equation (5.6) [5].

Lemma 5.1. *If $f \in L^p(0,T;L^2)$ for $1 < p < \infty$, problem (5.6) has a unique solution $u \in W^{1,p}(0,T;L^2) \cap L^p(0,T;\check{H}^2)$ given by (5.7), which satisfies*

$$\|u\|_{W^{1,p}(0,t;L^2)} + \|u\|_{L^p(0,t;\check{H}^2)} \le Q\|f\|_{L^p(0,t;L^2)}, \quad 0 < t \le T$$

where Q is independent of f, t and T.

5.2 Well-posedness and solution regularity

We prove well-posedness of problem (5.1) in the following Theorem.

Theorem 5.2. *[124] If $\Delta u_0 \in L^2$ and $f \in L^p(L^2)$ for $1 < p < \infty$, problem* (5.1) *has a unique solution $u \in W^{1,p}(L^2) \cap L^p(\check{H}^2)$ such that for some $Q = Q(\gamma^*, T, p)$*

$$\|u\|_{W^{1,p}(L^2)} + \|u\|_{L^p(\check{H}^2)} \le Q\Big(\|f\|_{L^p(L^2)} + \|\Delta u_0\|_{L^2}\Big).$$

Proof. We may assume $u_0 = 0$ by replacing u and f in problem (5.1) with $u - u_0$ and $f + \Delta u_0$, respectively. Let $\mathcal{X}_\lambda$ be the space $\mathcal{X} := \Big\{g \in W^{1,p}(L^2) : g(\boldsymbol{x}, 0) = 0\Big\}$ equipped with the equivalent norm $\|g\|_{\mathcal{X}_\lambda} := \Big\|e^{-\lambda t}\partial_t g\Big\|_{L^p(L^2)}$ for some $\lambda \ge 1$ [40]. We define a map $\mathcal{M} : \mathcal{X}_\lambda \to \mathcal{X}_\lambda$: for any $v \in \mathcal{X}_\lambda$, let $w = \mathcal{M}v$ be the solution of

$$\begin{aligned} \partial_t w(\boldsymbol{x}, t) - \Delta w(\boldsymbol{x}, t) = f(\boldsymbol{x}, t) - k\partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times (0, T]; \\ w(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, T]; \qquad w(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in \Omega. \end{aligned} \tag{5.9}$$

We apply the following relations

$$\begin{aligned} \int_0^T e^{-\lambda t} t^{-\gamma^*} dt \; &\le \lambda^{\gamma^*-1} \int_0^\infty e^{-s} s^{-\gamma^*} ds = \lambda^{\gamma^*-1}\Gamma(1-\gamma^*), \\ (t-s)^{-\gamma(\boldsymbol{x},s)} \; &= (t-s)^{-\gamma^*}(t-s)^{\gamma^*-\gamma(\boldsymbol{x},s)} \le \max\{1, T\}(t-s)^{-\gamma^*} \end{aligned} \tag{5.10}$$

and Young's convolution inequality to obtain

$$\begin{aligned} \left\| e^{-\lambda t} \partial_t^{\gamma(\boldsymbol{x},t)} v \right\|_{L^p(L^2)} &= \left\| e^{-\lambda t} \int_0^t \frac{\partial_s v(\boldsymbol{x},s)}{\Gamma(1-\gamma(\boldsymbol{x},s))(t-s)^{\gamma(\boldsymbol{x},s)}} ds \right\|_{L^p(L^2)} \\ &\leq Q \left\| \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\gamma^*}} e^{-\lambda s} \partial_s v(\boldsymbol{x},s) ds \right\|_{L^p(L^2)} \\ &\leq Q \left\| \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\gamma^*}} e^{-\lambda s} \left\| \partial_s v(\boldsymbol{x},s) \right\|_{L^2(\Omega)} ds \right\|_{L^p(0,T)} \\ &\leq Q \left\| e^{-\lambda t} t^{-\gamma^*} \right\|_{L^1(0,T)} \left\| e^{-\lambda t} \partial_t v \right\|_{L^p(L^2)} \leq Q \lambda^{\gamma^*-1} \left\| e^{-\lambda t} \partial_t v \right\|_{L^p(L^2)}. \end{aligned} \tag{5.11}$$

Therefore, $k\, \partial_t^{\gamma(\boldsymbol{x},t)} v \in L^p(0,T;L^2)$. By Lemma 5.1 problem (5.9) has a unique solution $w \in \mathcal{X}_\lambda$ and the mapping $\mathcal{M}: \mathcal{X}_\lambda \to \mathcal{X}_\lambda$ is well defined.

Let $w_1 = \mathcal{M} v_1$ and $w_2 = \mathcal{M} v_2$ for $v_1, v_2 \in \mathcal{X}_\lambda$, then $w = w_1 - w_2 \in \mathcal{X}_\lambda$ satisfies

$$\partial_t w(\boldsymbol{x},t) - \Delta w(\boldsymbol{x},t) = -k \partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t), \quad v := v_1 - v_2,$$

equipped with the zero initial and boundary conditions. Then (5.7) becomes

$$w(\boldsymbol{x},t) = -\int_0^t E(t-s)\Big(k\, \partial_s^{\gamma(\boldsymbol{x},s)} v(\boldsymbol{x},s)\Big) ds. \tag{5.12}$$

We differentiate (5.12) with respect to t and multiply the equation by $e^{-\lambda t}$ to get

$$\begin{aligned} e^{-\lambda t} \partial_t w(\boldsymbol{x},t) \; &= -k\, e^{-\lambda t} \partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t) \\ &\quad - e^{-\lambda t} \int_0^t \partial_t E(t-s)\Big(k\, \partial_s^{\gamma(\boldsymbol{x},s)} v(\boldsymbol{x},s)\Big) ds. \end{aligned} \tag{5.13}$$

We use (5.11) to bound the first term on the right-hand side, and use (5.8) and Laplace transform to evaluate the second term to get that for $0 < \varepsilon \ll 1$

$$\begin{aligned} &\mathcal{L}\Big[\int_0^t \partial_t E(t-s)\Big(k\, \partial_s^{\gamma(\boldsymbol{x},s)} v(\boldsymbol{x},s)\Big) ds \Big] \\ &= \mathcal{L}\Big[\int_0^t \partial_t \Big(\frac{1}{2\pi i} \int_{\Gamma_\theta} e^{z(t-s)} (z-\Delta)^{-1} dz \Big) \Big(k\, \partial_s^{\gamma(\boldsymbol{x},s)} v(\boldsymbol{x},s)\Big) ds \Big] \\ &= \mathcal{L}\Big[\int_0^t \Big(\frac{1}{2\pi i} \int_{\Gamma_\theta} e^{z(t-s)} z (z-\Delta)^{-1} dz \Big) \Big(k\, \partial_s^{\gamma(\boldsymbol{x},s)} v(\boldsymbol{x},s)\Big) ds \Big] \\ &= \mathcal{L}\Big[\frac{1}{2\pi i} \int_{\Gamma_\theta} e^{zt} z (z-\Delta)^{-1} dz \Big] \mathcal{L}\Big(k\, \partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t)\Big) \\ &= z(z-\Delta)^{-1} \mathcal{L}\Big(k \partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t)\Big) = \Big(z^{1-\varepsilon}(z-\Delta)^{-1}\Big)\Big(z^{\varepsilon} \mathcal{L}(k \partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t))\Big). \end{aligned} \tag{5.14}$$

We take the inverse Laplace transform of (5.14) and use (5.4)-(5.5) to get

$$\begin{aligned}
&\int_0^t \partial_t E(t-s)\Big(k\,\partial_s^{\gamma(\boldsymbol{x},s)} v(\boldsymbol{x},s)\Big)ds \\
&= \mathcal{L}^{-1}\Big[\Big(z^{1-\varepsilon}(z-\Delta)^{-1}\Big)\Big(z^{\varepsilon}\mathcal{L}(k\partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t))\Big)\Big] \\
&= \mathcal{L}^{-1}\Big[\Big(z^{1-\varepsilon}(z-\Delta)^{-1}\Big)\mathcal{L}\Big(k\,{}^R\partial_t^{\varepsilon}\,\partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t)\Big)\Big] \\
&= \Big[\mathcal{L}^{-1}\Big(z^{1-\varepsilon}(z-\Delta)^{-1}\Big)\Big] * \Big(k\,{}^R\partial_t^{\varepsilon}\,\partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t)\Big) \\
&= \Big[\frac{1}{2\pi i}\int_{\Gamma_\theta} z^{1-\varepsilon}(z-\Delta)^{-1}e^{zt}dz\Big] * \Big(k\,{}^R\partial_t^{\varepsilon}\,\partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t)\Big) \\
&= \int_0^t \Big[\frac{1}{2\pi i}\int_{\Gamma_\theta} z^{1-\varepsilon}(z-\Delta)^{-1}e^{z(t-s)}dz\Big]\Big(k\,{}^R\partial_s^{\varepsilon}\,\partial_s^{\gamma(\boldsymbol{x},s)} v(\boldsymbol{x},s)\Big)ds.
\end{aligned} \tag{5.15}$$

We use (5.3) to bound the integral in the square bracket by

$$\Big\|\frac{1}{2\pi i}\int_{\Gamma_\theta} z^{1-\varepsilon}(z-\Delta)^{-1}e^{z(t-s)}dz\Big\|_{L^2(\Omega)} \le \frac{Q}{(t-s)^{1-\varepsilon}}.$$

To bound the second integrand, we directly evaluate

$$\begin{aligned}
&{}_0I_t^{1-\varepsilon}\partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t) \\
&\quad = \frac{1}{\Gamma(1-\varepsilon)}\int_0^t \frac{1}{(t-y)^{\varepsilon}}\int_0^y \frac{1}{\Gamma(1-\gamma(\boldsymbol{x},\theta))}\frac{\partial_\theta v(\boldsymbol{x},\theta)}{(y-\theta)^{\gamma(\boldsymbol{x},\theta)}}d\theta dy \\
&\quad = \frac{1}{\Gamma(1-\varepsilon)}\int_0^t \frac{\partial_\theta v(\boldsymbol{x},\theta)}{\Gamma(1-\gamma(\boldsymbol{x},\theta))}\int_\theta^t \frac{1}{(t-y)^{\varepsilon}(y-\theta)^{\gamma(\boldsymbol{x},\theta)}}dy d\theta \\
&\quad = \int_0^t \frac{(t-\theta)^{1-\varepsilon-\gamma(\boldsymbol{x},\theta)}B(1-\gamma(\boldsymbol{x},\theta),1-\varepsilon)}{\Gamma(1-\varepsilon)\Gamma(1-\gamma(\boldsymbol{x},\theta))}\partial_\theta v(\boldsymbol{x},\theta)d\theta \\
&\quad = \int_0^t \frac{(t-\theta)^{1-\varepsilon-\gamma(\boldsymbol{x},\theta)}}{\Gamma(2-\varepsilon-\gamma(\boldsymbol{x},\theta))}\partial_\theta v(\boldsymbol{x},\theta)d\theta.
\end{aligned} \tag{5.16}$$

We use (5.16) to bound the following term on the right-hand side of (5.15) by

$$\begin{aligned}
\Big|{}^R\partial_t^{\varepsilon}\,\partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t)\Big| &= \Big|\partial_t\, {}_0I_t^{1-\varepsilon}\,\partial_t^{\gamma(\boldsymbol{x},t)} v(\boldsymbol{x},t)\Big| \\
&= \int_0^t \frac{(t-\theta)^{-\varepsilon-\gamma(\boldsymbol{x},\theta)}\partial_\theta v(\boldsymbol{x},\theta)}{\Gamma(1-\varepsilon-\gamma(\boldsymbol{x},\theta))}d\theta \le Q\int_0^t \frac{|\partial_\theta v(\boldsymbol{x},\theta)|}{(t-\theta)^{\varepsilon+\gamma^*}}d\theta.
\end{aligned} \tag{5.17}$$

We combine (5.14)-(5.17) to bound the second term on the right-hand side of (5.13)

$$\begin{aligned}&\left\| e^{-\lambda t} \int_0^t \partial_t E(t-s)\Big(k\, \partial_s^{\gamma(\boldsymbol{x},s)} v(\boldsymbol{x},s)\Big) ds \right\|_{L^2} \\ &\quad \le Q e^{-\lambda t} \int_0^t \frac{1}{(t-s)^{1-\varepsilon}} \int_0^s \frac{\|\partial_\theta v(\cdot,\theta)\|_{L^2}}{(s-\theta)^{\varepsilon+\gamma^*}} d\theta ds \\ &\quad \le Q e^{-\lambda t} \int_0^t \frac{\big\|\partial_\theta v(\cdot,\theta)\big\|_{L^2}}{(t-\theta)^{\gamma^*}} d\theta = Q \int_0^t \frac{e^{-\lambda(t-\theta)} \big\| e^{-\lambda\theta} \partial_\theta v(\cdot,\theta)\big\|_{L^2}}{(t-\theta)^{\gamma^*}} d\theta.\end{aligned}$$

We take the $\|\cdot\|_{L^p(0,T)}$ norm of the preceding inequality and use (5.10) and Young's convolution inequality to obtain

$$\begin{aligned}&\left\| e^{-\lambda t} \int_0^t \partial_t E(t-s)\Big(k\, \partial_s^{\gamma(\boldsymbol{x},s)} v(\boldsymbol{x},s)\Big) ds \right\|_{L^p(L^2)} \\ &\quad \le Q \left\| \Big(e^{-\lambda t} t^{-\gamma^*}\Big) * \big\| e^{-\lambda t} \partial_t v(\cdot,t)\big\|_{L^2} \right\|_{L^p(0,T)} \le Q \lambda^{\gamma^*-1} \big\| e^{-\lambda t} \partial_t v \big\|_{L^p(L^2)}.\end{aligned} \tag{5.18}$$

We combine estimates (5.11) and (5.18) to bound the right-hand side of (5.13) by

$$\|w\|_{\mathcal{X}_\lambda} \le Q \lambda^{\gamma^*-1} \|v\|_{\mathcal{X}_\lambda}.$$

Choosing a sufficiently large λ ensures that the mapping $\mathcal{M} : \mathcal{X}_\lambda \to \mathcal{X}_\lambda$ is a contraction. By the Banach fixed point theorem, $\mathcal{M}$ has a unique fixed point $w \in \mathcal{X}_\lambda$, that is, (5.1) with $u_0 = 0$, and consequently (5.1), has a unique solution in $\mathcal{X}$.

Let $w = \mathcal{M}w$ be the fixed point. Problem (5.9) becomes

$$\begin{aligned}&\partial_t w(\boldsymbol{x},t) - \Delta w(\boldsymbol{x},t) = f(\boldsymbol{x},t) - k \partial_t^{\gamma(\boldsymbol{x},t)} w(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega \times (0,T]; \\ &w(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t) \in \partial\Omega \times (0,T]; \quad w(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega.\end{aligned}$$

We multiply $e^{-\lambda t}$ on both sides to rewrite the equation as

$$\partial_t(e^{-\lambda t} w) - \Delta(e^{-\lambda t} w) = -\lambda e^{-\lambda t} w + e^{-\lambda t} f - k e^{-\lambda t} \partial_t^{\gamma(\boldsymbol{x},t)} w.$$

We apply (5.11) and Lemma 5.1 to conclude that for any $0 < \bar{t} \le T$

$$\begin{aligned}&\|e^{-\lambda t} w\|_{W^{1,p}(0,\bar{t};L^2)} \\ &\quad \le Q\Big(\|f\|_{L^p(0,\bar{t};L^2)} + \lambda \|e^{-\lambda t} w\|_{L^p(0,\bar{t};L^2)} + \big\| e^{-\lambda t} \partial_t^{\gamma(\boldsymbol{x},t)} w \big\|_{L^p(0,\bar{t};L^2)} \Big) \\ &\quad \le Q\Big(\|f\|_{L^p(0,\bar{t};L^2)} + \lambda \|e^{-\lambda t} w\|_{L^p(0,\bar{t};L^2)} + \lambda^{\gamma^*-1} \big\| e^{-\lambda t} \partial_t w \big\|_{L^p(0,\bar{t};L^2)} \Big).\end{aligned} \tag{5.19}$$

Choosing λ in (5.19) sufficiently large yields

$$\left\|e^{-\lambda t}\partial_t w\right\|_{L^p(0,\bar{t};L^2)} \le Q\lambda\|e^{-\lambda t}w\|_{L^p(0,\bar{t};L^2)} + Q\|f\|_{L^p(0,\bar{t};L^2)}, \tag{5.20}$$

where Q is independent of λ. We bound the first term on the right-hand side by

$$\begin{aligned}\left\|e^{-\lambda t}w\right\|^p_{L^p(0,\bar{t};L^2)} &= \int_0^{\bar{t}}\left\|e^{-\lambda t}\int_0^t \partial_s w(\cdot,s)ds\right\|^p_{L^2}dt\\ &\le Q\int_0^{\bar{t}}\left(\int_0^t \left\|e^{-\lambda s}\partial_s w(\cdot,s)\right\|_{L^2}ds\right)^p dt\\ &\le Q\int_0^{\bar{t}}\int_0^t \left\|e^{-\lambda s}\partial_y w(\cdot,s)\right\|^p_{L^2}dsdt = Q\int_0^{\bar{t}}\|e^{-\lambda s}\partial_s w\|^p_{L^p(0,t;L^2)}dt.\end{aligned}$$

We plug this estimate into (5.20) we find

$$\left\|e^{-\lambda t}\partial_t w\right\|^p_{L^p(0,\bar{t};L^2)} \le Q\lambda^p\int_0^{\bar{t}}\|e^{-\lambda t}\partial_t w\|^p_{L^p(0,s;L^2)}ds + Q\|f\|^p_{L^p(0,T;L^2)}, \quad 0<\bar{t}\le T.$$

Applying Gronwall's inequality yields

$$\left\|e^{-\lambda t}\partial_t w\right\|_{L^p(0,\bar{t};L^2)} \le Qe^{Q\lambda^p T}\|f\|_{L^p(0,T;L^2)}, \quad 0<\bar{t}\le T. \tag{5.21}$$

Let u be the solution to problem (5.1). Then $w(\boldsymbol{x},t) := u(\boldsymbol{x},t) - u_0(\boldsymbol{x})$ satisfies

$$\begin{aligned}\partial_t w(\boldsymbol{x},t) - \Delta w(\boldsymbol{x},t) = f(\boldsymbol{x},t) + \Delta u_0(\boldsymbol{x}) - k\partial_t^{\gamma(\boldsymbol{x},t)}w(\boldsymbol{x},t),&\\ (\boldsymbol{x},t)\in\Omega\times(0,T];&\end{aligned} \tag{5.22}$$

$$w(\boldsymbol{x},t)=0, \quad (\boldsymbol{x},t)\in\partial\Omega\times(0,T]; \quad w(\boldsymbol{x},0)=0, \quad \boldsymbol{x}\in\Omega.$$

We apply estimate (5.21) to problem (5.22) to get

$$\|w\|_{W^{1,p}(0,T;L^2)} \le Q\Big(\|f\|_{L^p(0,T;L^2)} + \|\Delta u_0\|_{L^2}\Big)$$

where Q depends on λ, which yields

$$\|u\|_{W^{1,p}(0,T;L^2)} \le Q\Big(\|f\|_{L^p(0,T;L^2)} + \|\Delta u_0\|_{L^2}\Big).$$

Then we apply this equation, (5.1) and similar estimate as (5.11) to find

$$\begin{aligned}\|u\|_{L^p(\check{H}^2)} &\le Q\|\Delta u\|_{L^p(L^2)} = Q\left\|\partial_t u + k\partial_t^{\gamma(\boldsymbol{x},t)}u - f\right\|_{L^p(L^2)}\\ &\le Q\Big(\|u\|_{W^{1,p}(L^2)} + \|f\|_{L^p(L^2)}\Big) \le Q\Big(\|f\|_{L^p(L^2)} + \|\Delta u_0\|_{L^2}\Big).\end{aligned}$$

We thus complete the proof. □

To estimate $\partial_{tt}u$ for the analysis of the numerical approximation to problem (5.1), we use (5.7) to express the solution w to problem (5.22) as follows

$$w(\boldsymbol{x},t)=\int_0^t E(t-s)\Big(\Delta u_0+f(\boldsymbol{x},s)\Big)\mathrm{d}s-\int_0^t E(t-s)(k\,\partial_s^{\gamma(\boldsymbol{x},s)}w(\boldsymbol{x},s))\mathrm{d}s. \quad (5.23)$$

Then the estimate of $\partial_{tt}w$ could be performed by differentiating (5.23) twice in time and then applying similar techniques in the proof of Theorem 5.2. We thus present the result without proof and refer readers to [124] for more details.

Theorem 5.3. *Suppose* $\gamma\in W^{2,\infty}(L^\infty)$, $\Delta u_0\in \check{H}^{\frac{d}{2}+\varepsilon}$ *and* $f\in W^{1,1}\big(\check{H}^{\frac{d}{2}+\varepsilon}\big)$ *for* $0<\varepsilon\ll 1$. *The regularity estimate holds*

$$\|u\|_{W^{2,1}(L^\infty)}\le Q\Big(\|\Delta u_0\|_{\check{H}^{\frac{d}{2}+\varepsilon}}+\|f\|_{W^{1,1}(\check{H}^{\frac{d}{2}+\varepsilon})}\Big)$$

with $Q=Q(\varepsilon,\gamma^*,\|\gamma\|_{W^{2,\infty}(L^\infty)},k,T)$.

If further $\gamma\in C^2([0,T];L^\infty)$, $\Delta u_0\in \check{H}^{2(1-\gamma_0)+\frac{d}{2}+\varepsilon}$, $f(\boldsymbol{x},0)\in \check{H}^{2(1-\gamma_0)+\frac{d}{2}+\varepsilon}$ *and* $f\in C^1\big([0,T];\check{H}^{\frac{d}{2}+\varepsilon}\big)$ *for* $0<\varepsilon\ll 1$, *then* $u\in C^2((0,T];L^\infty)$ *and the pointwise-in-time estimate holds for* $t\in(0,T]$

$$\begin{aligned}\|\partial_{tt}u(\cdot,t)\|_{L^\infty(\Omega)} &\le Qt^{-\gamma_0}\Big(\|\Delta u_0\|_{\check{H}^{2(1-\gamma_0)+\frac{d}{2}+\varepsilon}}+\|f(\cdot,0)\|_{\check{H}^{2(1-\gamma_0)+\frac{d}{2}+\varepsilon}}\\ &\quad+\|f\|_{C^1([0,T];\check{H}^{\frac{d}{2}+\varepsilon})}\Big),\quad \gamma_0:=\|\gamma(\boldsymbol{x},0)\|_{L^\infty},\end{aligned}$$

with $Q=Q(\varepsilon,\gamma^*,\|\gamma\|_{W^{2,\infty}(L^\infty)},k,T)$. *Further, if* $\gamma(\boldsymbol{x},0)\equiv 0$, *the estimate is improved to the following global regularity estimate*

$$\begin{aligned}\|u\|_{C^2([0,T];L^\infty)} &\le Q\Big(\|\Delta u_0\|_{\check{H}^{2(1-\gamma_0)+\frac{d}{2}+\varepsilon}}\\ &\quad+\|f(\cdot,0)\|_{\check{H}^{2(1-\gamma_0)+\frac{d}{2}+\varepsilon}}+\|f\|_{C^1([0,T];\check{H}^{\frac{d}{2}+\varepsilon})}\Big).\end{aligned}$$

5.3 Discretization and numerical experiments

We follow [124] to present and analyze a discrete-in-time scheme to model (5.1). Define a uniform partition on $[0,T]$ by $t_n:=n\tau$ for $\tau:=T/N$ and $0\le n\le N$. Let

$u_n := u(\boldsymbol{x}, t_n)$, we discretize $\partial_t u$ and $\partial_t^{\gamma(\boldsymbol{x},t)} u$ at $t = t_n$ for $1 \le n \le N$ by

$$
\begin{aligned}
\partial_t u(\boldsymbol{x}, t_n) &= \delta_\tau u_n + R_n := \frac{u_n - u_{n-1}}{\tau} + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \partial_{tt} u(\boldsymbol{x}, t)(t - t_{n-1})\mathrm{d}t, \\
\partial_t^{\gamma(\boldsymbol{x},t_n)} u(\boldsymbol{x}, t_n) &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s)\mathrm{d}s}{\Gamma(1-\gamma(\boldsymbol{x}, s))(t_n - s)^{\gamma(\boldsymbol{x},s)}} \\
&= \sum_{k=1}^{n} \Bigg[\int_{t_{k-1}}^{t_k} \frac{\delta_\tau u_k \mathrm{d}s}{\Gamma(1-\gamma(\boldsymbol{x}, t_k))(t_n - s)^{\gamma(\boldsymbol{x},t_k)}} \\
&\quad + \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s)\mathrm{d}s}{\Gamma(1-\gamma(\boldsymbol{x}, s))(t_n - s)^{\gamma(\boldsymbol{x},s)}} - \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s)\mathrm{d}s}{\Gamma(1-\gamma(\boldsymbol{x}, t_k))(t_n - s)^{\gamma(\boldsymbol{x},t_k)}} \\
&\quad + \int_{t_{k-1}}^{t_k} \frac{(\partial_s u(\boldsymbol{x}, s) - \delta_\tau u_k)\mathrm{d}s}{\Gamma(1-\gamma(\boldsymbol{x}, t_k))(t_n - s)^{\gamma(\boldsymbol{x},t_k)}} \Bigg] = \delta_\tau^{\gamma(\boldsymbol{x},t_n)} u_n + F_n + G_n.
\end{aligned}
\tag{5.24}
$$

Here $\delta_\tau^{\gamma(\boldsymbol{x},t_n)} u_n$, F_n and G_n are defined by

$$
\begin{aligned}
\delta_\tau^{\gamma(\boldsymbol{x},t_n)} u_n &:= \sum_{k=1}^{n} c_{n,k}(u_k - u_{k-1}), \\
c_{n,k} &:= \frac{1}{\tau}\int_{t_{k-1}}^{t_k} \frac{(t_n - s)^{-\gamma(\boldsymbol{x},t_k)}}{\Gamma(1-\gamma(\boldsymbol{x}, t_k))}\mathrm{d}s = \frac{(t_n - t_{k-1})^{1-\gamma(\boldsymbol{x},t_k)} - (t_n - t_k)^{1-\gamma(\boldsymbol{x},t_k)}}{\Gamma(2-\gamma(\boldsymbol{x}, t_k))\tau}, \\
F_n &:= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left[\frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\gamma(\boldsymbol{x}, s))(t_n - s)^{\gamma(\boldsymbol{x},s)}} - \frac{\partial_s u(\boldsymbol{x}, s)}{\Gamma(1-\gamma(\boldsymbol{x}, t_k))(t_n - s)^{\gamma(\boldsymbol{x},t_k)}} \right]\mathrm{d}s, \\
G_n &:= \sum_{k=1}^{n} G_{n,k} := \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{\partial_s u(\boldsymbol{x}, s) - \delta_\tau u_k}{\Gamma(1-\gamma(\boldsymbol{x}, t_k))(t_n - s)^{\gamma(\boldsymbol{x},t_k)}}\mathrm{d}s \\
&= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{1}{\tau\Gamma(1-\gamma(\boldsymbol{x}, t_k))(t_n - s)^{\gamma(\boldsymbol{x},t_k)}} \left[\int_{t_{k-1}}^{t_k} \int_z^s \partial_{\theta\theta} u(\boldsymbol{x}, \theta)\mathrm{d}\theta\mathrm{d}z \right]\mathrm{d}s.
\end{aligned}
$$

We plug (5.24) into (5.1) to derive a reference equation for problem (5.1)

$$
\delta_\tau u_n + k\, \delta_\tau^{\gamma(\boldsymbol{x},t_n)} u_n - \Delta u_n = f(\boldsymbol{x}, t_n) - \Big(k(F_n + G_n) + R_n\Big), \quad 1 \le n \le N.
$$

We drop the last term on the right-hand side to obtain an L-1 time-discrete scheme for problem (5.1): find $U_n = U_n(\boldsymbol{x})$ for $n = 1, 2, \ldots, N$, with $U_0 := u_0$, such that

$$
\delta_\tau U_n + k\, \delta_\tau^{\gamma(\boldsymbol{x},t_n)} U_n - \Delta U_n = f(\boldsymbol{x}, t_n). \tag{5.25}
$$

The truncation errors can be analyzed by similar techniques in Section 3.3.1 and the optimal-order error estimate of the semi-discrete in time scheme (5.25) can be carried out in parallel with the proof of Theorem 4.7. We thus present the results without proof. More detailed derivations can be found in [124].

Lemma 5.4. *If* $\gamma \in W^{2,\infty}(L^\infty)$, $\Delta u_0 \in \check{H}^{\frac{d}{2}+\varepsilon}$ *and* $f \in W^{1,1}(\check{H}^{\frac{d}{2}+\varepsilon})$ *for some* $0 < \varepsilon \ll 1$, *then the following estimate holds*

$$\|R\|_{\hat{L}^1(L^\infty)} + \|F\|_{\hat{L}^1(L^\infty)} + \|G\|_{\hat{L}^1(L^\infty)} \leq QQ_0\tau, \quad \|R\|_{\hat{L}^1(L^\infty)} := \tau \sum_{n=1}^{N} \|R_n\|_{L^\infty}.$$

Here $Q := Q(\varepsilon, \gamma^*, k, \|\gamma\|_{W^{2,\infty}(L^\infty)}, T)$ *and* $Q_0 := \|\Delta u_0\|_{\check{H}^{\frac{d}{2}+\varepsilon}} + \|f\|_{W^{1,1}(\check{H}^{\frac{d}{2}+\varepsilon})}$.

Theorem 5.5. *If* $\gamma \in W^{2,\infty}(L^\infty)$, $\Delta u_0 \in \check{H}^{\frac{d}{2}+\varepsilon}$ *and* $f \in W^{1,1}(\check{H}^{\frac{d}{2}+\varepsilon})$ *for some* $0 < \varepsilon \ll 1$, *the optimal-order error estimate holds for scheme* (5.25)

$$\|U - u\|_{\hat{L}^\infty(L^\infty)} := \max_{1 \leq n \leq N} \|U_n - u_n\|_{L^\infty} \leq Q\tau\Big(\|\Delta u_0\|_{\check{H}^{\frac{d}{2}+\varepsilon}} + \|f\|_{W^{1,1}(\check{H}^{\frac{d}{2}+\varepsilon})}\Big).$$

Here $Q = Q(\varepsilon, k, \gamma^*, \|\gamma\|_{W^{2,\infty}(L^\infty)}, T)$.

We then carry out numerical experiments to investigate the temporal convergence of the approximation (5.25) to problem (5.1). In numerical experiments we assume a rectangular domain $\Omega = (0,1)^d$, and apply the second order center difference scheme for $-\Delta$ under a uniform spatial partition with mesh size h. We take h small such that the errors of the spatial discretization can be neglected. We measure the error $\|U - u\|_{\hat{L}^\infty(L^\infty)}$ and fit the temporal convergence rate κ.

Example 1 Let $\Omega = (0,1)$, $[0,T] = [0,1]$, $k = 5$, $f = 0$, $u_0(x) = \sin(\pi x)$, and the variable order $\gamma(x,t) = \zeta(x)\eta(t)$ where

$$\zeta(x) = 1 + \frac{1}{10}\sin\left(\frac{\pi x}{2}\right), \quad \eta(t) = \iota_T + (\iota_0 - \iota_T)\left[1 - \frac{t}{T} - \frac{1}{2\pi}\sin\left(2\pi\Big(1 - \frac{t}{T}\Big)\right)\right]$$

and

$$\text{(i) } \iota_0 = 0,\ \iota_T = 0.3; \quad \text{(ii) } \iota_0 = 0.4,\ \iota_T = 0.3; \quad \text{(iii) } \iota_0 = 0.8,\ \iota_T = 0.3. \tag{5.26}$$

As a closed-form analytical solution is not available, we use a numerical solution computed with $\tau = 2^{-11}$ and $h = 2^{-5}$ as the reference solution to test the temporal convergence of the scheme. We present the numerical results in Table 5.1, which show the first-order temporal convergence rate of the scheme as proved in Theorem 5.5.

Table 5.1 Convergence rate of scheme (5.25) in Example 1 with γ given in (5.26) [124, Table 1].

τ	(i)	κ	(ii)	κ	(iii)	κ
1/32	3.34E-04		1.79E-03		5.28E-03	
1/48	2.27E-04	0.95	1.17E-03	1.05	3.45E-03	1.05
1/64	1.71E-04	0.98	8.63E-04	1.05	2.55E-03	1.05
1/80	8.44E-05	1.02	4.13E-04	1.06	1.22E-03	1.07

Example 2. We use scheme (5.25) to simulate the hidden-memory variable-order tFDE (5.1) in three space dimensions. The data are given as follows: $\Omega = (0,1)^3$, $[0,T] = [0,1]$, $\kappa(\boldsymbol{x}) = 5$, $f = 0$, $u_0(x_1,x_2,x_3) = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)$, and the variable order $\gamma(\boldsymbol{x},t) = \zeta(\boldsymbol{x})\eta(t)$ is given by

$$\zeta(x_1,x_2,x_3) = \Big(1 - \frac{x_1^2}{5}\Big)\Big(1 + \frac{\sin(\pi x_2)}{10}\Big)\exp\Big(\frac{x_3}{10}\Big)$$

$$\text{(iv) } \eta(t) = \frac{t}{5} + \frac{3}{10}; \qquad \text{(v) } \eta(t) = -\frac{t^2}{5} + \frac{3}{5}. \tag{5.27}$$

We use the numerical solution computed with $\tau = 2^{-9}$ and $h = 1/24$ as the reference solution to test the temporal convergence rate of scheme (5.25). We present the numerical results in Table 5.2, which again show the first-order temporal convergence rate of the scheme (5.25) as proved in Theorem 5.5.

Table 5.2 Convergence of scheme (5.25) in Example 2 with γ given in (5.27) [124, Table 2].

τ	(iv)	κ	(v)	κ
1/32	8.19E-05		1.08E-03	
1/48	5.72E-05	0.89	6.78E-04	1.14
1/64	4.30E-05	0.99	4.85E-04	1.16
1/80	3.39E-05	1.07	3.72E-04	1.20

Chapter 6

Variable-order space-fractional PDEs

SFDEs are widely used to model the anomalous diffusion in heterogeneous media and thus are extensively investigated [1, 2, 21, 67, 68, 83, 95, 108, 109, 110]. In this chapter, we study the following variable-order sFDE, which serves as a variable-order extension of the steady-state counterpart of the mixed diffusion model (cf. [14, Equation 29]) that consists of both a second-order spatial derivative term modeling the Fickian diffusive transport and a space-fractional derivative term modeling the superdiffusive transport

$$-\partial_{xx}u(x) - k\, {}_0\partial_x^{\alpha(x)}u(x) = f(x),\ x \in (0,1); \quad u(0) = u(1) = 0 \tag{6.1}$$

where $1 \leq \alpha(x) < 2$ and $k \geq 0$ is the fractional diffusivity and the variable order $\alpha(x)$ satisfies the following assumptions:

Assumption C $\alpha \in C^1[0,T]$ and $1 \leq \alpha(t) \leq \alpha^* < 2$ on $[0,T]$ for some $1 < \alpha^* < 2$.

In this chapter we follow [113] to carry out mathematical and numerical analysis to model (6.1).

6.1 Well-posedness and solution regularity

By the definition (2.5) of ${}_0\partial_x^{\alpha(x)}$, we decompose the equation (6.1) into the following system via the substitution $v(x) := \partial_{xx}u(x)$

$$v(x) + k\, {}_0I_x^{2-\alpha(x)}v(x) = -f(x), \tag{6.2}$$

$$u(x) = v * x - x \cdot (v * x|_{x=1}), \tag{6.3}$$

where $*$ represents the convolution on $[0, x]$. Therefore, the main task is to analyze the first integral equation in (6.1). Since the well-posedness of this integral and the estimate of $\partial_x v = \partial_x^3 u$ can be analyzed by similar (and simpler) techniques as those in Chapter 3, we cite results from [113] and omit their proofs, and focus on the estimate of $\partial_{xx} v = \partial_x^4 u$, which has stronger singularities not encountered in previous chapters.

Theorem 6.1. *Under Assumption C, (6.2) has a unique solution $v \in C[0,1]$ and*

$$\|v\|_{C[0,1]} \leq Q\|f\|_{C[0,1]}, \quad Q = Q(\alpha^*, k). \tag{6.4}$$

That is, (6.1) has a unique solution $u \in C^2[0,1]$ such that

$$\|u\|_{C^2[0,1]} \leq Q\|f\|_{C[0,1]}, \quad Q = Q(\alpha^*, k).$$

Theorem 6.2. *Suppose $f \in C^1[0,1]$ and the Assumption C holds. Then the solution u to* (6.1) *belongs to $C^3(0,1]$ and the following estimate holds*

$$|\partial_x^3 u| \leq Q_* \|f\|_{C^1[0,1]} x^{1-\alpha(0)}, \quad 0 < x \leq 1, \tag{6.5}$$

where $Q_* = Q_*(\alpha^*, k, \|\alpha\|_{C^1[0,1]})$. If $\alpha(0) = 1$ then $u \in C^3[0,1]$ and

$$\|\partial_x^3 u\|_{C[0,1]} \leq Q_* \|f\|_{C^1[0,1]}. \tag{6.6}$$

Theorem 6.3. *[113] Suppose $f, \alpha \in C^2[0,1]$ and Assumption C holds. If $\alpha(0) > 1$, then $u \in C^2[0,1] \cap C^4(0,1]$ and*

$$|\partial_x^4 u| \leq Q_{**} \|f\|_{C^2[0,1]} x^{-\alpha(0)}, \quad 0 < x \leq 1 \tag{6.7}$$

where $Q_{**} = Q_{**}\big(\alpha^*, \|\alpha\|_{C^2[0,1]}, k\big)$. If $\alpha(0) = 1$ and $\alpha'(0) = 0$, then $u \in C^4[0,1]$ and

$$\|\partial_x^4 u\|_{C[0,1]} \leq Q_{**} \|f\|_{C^2[0,1]}.$$

Proof. We first consider the case of $\alpha(0) > 1$. We apply integration by parts for ${}_0I_x^{2-\alpha(x)}v(x)$ and then differentiate (6.2) twice to obtain

$$\begin{aligned}
v''(x) &= \Big(\frac{-k}{\Gamma(3-\alpha(x))}\Big)''\Big(x^{2-\alpha(x)}v(0) + \int_0^x v'(s)(x-s)^{2-\alpha(x)}ds\Big) \\
&\quad -\Big(\frac{k}{\Gamma(3-\alpha(x))}\Big)'\Big[\Big(x^{2-\alpha(x)}v(0) + \int_0^x v'(s)(x-s)^{2-\alpha(x)}ds\Big)' \\
&\quad - \Big(\alpha'(x)x^{2-\alpha(x)}\ln x - \frac{2-\alpha(x)}{x^{\alpha(x)-1}}\Big)v(0) \\
&\quad - \int_0^x v'(s)\Big((x-s)^{2-\alpha(x)}\alpha'(x)\ln(x-s) - \frac{2-\alpha(x)}{(x-s)^{\alpha(x)-1}}\Big)ds\Big] \\
&\quad + \frac{k(x)}{\Gamma(3-\alpha(x))}\Big[\Big(\alpha'(x)x^{2-\alpha(x)}\ln x - \frac{2-\alpha(x)}{x^{\alpha(x)-1}}\Big)v(0) \\
&\quad - \int_0^x v'(s)\Big((x-s)^{2-\alpha(x)}\alpha'(x)\ln(x-s) - \frac{2-\alpha(x)}{(x-s)^{\alpha(x)-1}}\Big)ds\Big]' - f''(x).
\end{aligned} \tag{6.8}$$

We observe that the tenth and the twelfth term on the right-hand side have the strongest singularity. Hence, we focus on the estimates of these two terms and omit the estimates of the rest. We recall $\alpha(0) > 1$ and apply (6.4) and the fact that

$$\Big(x^{1-\alpha(x)}\Big)' = -x^{1-\alpha(x)}\alpha'(x)\ln x + \frac{1-\alpha(x)}{x^{\alpha(x)}}$$

to bound the tenth term on the right-hand side for $x \in (\varepsilon, 1]$ by

$$\Big|\frac{k}{\Gamma(3-\alpha(x))}\Big(\frac{2-\alpha(x)}{x^{\alpha(x)-1}}\Big)'v(0)\Big| \le \frac{Q|f(0)|}{x^{\alpha(x)}} = \frac{Q|f(0)|}{x^{\alpha(0)}x^{\alpha(x)-\alpha(0)}} \le \frac{Q|f(0)|}{x^{\alpha(0)}} \le \frac{Q|f(0)|}{\varepsilon^{\alpha(0)}}.$$

We decompose the twelfth term on the right-hand side of (6.8) for $x \in (\varepsilon, 1]$ as

$$\begin{aligned}
&\frac{k\alpha'(x)}{\Gamma(3-\alpha(x))}\int_0^x \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}} - \frac{k}{\Gamma(2-\alpha(x))}\Big(\int_0^x \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}}\Big)' \\
&\quad = \frac{k\alpha'(x)}{\Gamma(3-\alpha(x))}\int_0^x \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}} - \frac{k}{\Gamma(2-\alpha(x))}\Big(\int_0^{\varepsilon/2} \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}} \\
&\qquad + \int_{\varepsilon/2}^{\varepsilon} \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}} + \int_{\varepsilon}^{x} \frac{v'(s)ds}{(x-s)^{\alpha(x)-1}}\Big)' =: \sum_{i=1}^4 J_i.
\end{aligned}$$

We use (6.5) to bound J_1 by

$$\begin{aligned}
|J_1| &\le Q\|f\|_{C^1[0,1]}\int_0^x s^{1-\alpha(0)}(x-s)^{1-\alpha(x)}ds \\
&= Q\|f\|_{C^1[0,1]}B(2-\alpha(0), 2-\alpha(x))x^{3-\alpha(0)-\alpha(x)} \le Q\|f\|_{C^1[0,1]}x^{1-\alpha(0)}.
\end{aligned}$$

Note that the kernel in J_2 has no singularity since $x \in [\varepsilon, 1]$ and $s \in [0, \varepsilon/2]$. We bound $v'(s)$ in the integral by (6.5) and use the fact

$$\Big|(x-\varepsilon/2)^{-\int_{\varepsilon/2}^{x}\alpha'(\xi)d\xi}\Big| \le (x-\varepsilon/2)^{-\|\alpha\|_{C^1[0,1]}(x-\varepsilon/2)} \le Q$$

to bound J_2 by

$$\begin{aligned}
|J_2| &= \Big|\frac{k(x)}{\Gamma(2-\alpha(x))}\int_0^{\frac{\varepsilon}{2}} v'(s)\Big[\frac{-\alpha'(x)\ln(x-s)}{(x-s)^{\alpha(x)-1}} + \frac{1-\alpha(x)}{(x-s)^{\alpha(x)}}\Big]ds\Big| \\
&= \Big|\frac{k(x)}{\Gamma(2-\alpha(x))}\int_0^{\frac{\varepsilon}{2}} v'(s)\Big[\frac{-\alpha'(x)(x-s)\ln(x-s)+1-\alpha(x)}{(x-s)^{\alpha(x)}}\Big]ds\Big| \\
&\le Q\|f\|_{C^1[0,1]}(x-\varepsilon/2)^{-\alpha(x)}\int_0^{\frac{\varepsilon}{2}} s^{1-\alpha(0)}ds \\
&\le Q\|f\|_{C^1[0,1]}\varepsilon^{2-\alpha(0)}(x-\varepsilon/2)^{-\alpha(\varepsilon/2)-\int_{\varepsilon/2}^{x}\alpha'(\xi)d\xi} \\
&\le Q\|f\|_{C^1[0,1]}\varepsilon^{2-\alpha(0)}(x-\varepsilon/2)^{-\alpha(\varepsilon/2)} \le Q\|f\|_{C^1[0,1]}\varepsilon^{2-\alpha(0)-\alpha(\varepsilon/2)}.
\end{aligned}$$

The kernels in J_3 and J_4 may be singular, hence one cannot interchange the order of differentiation and integration directly. For $s < x$ with $x \in (\varepsilon, 1]$, we have

$$\begin{aligned}
&\Big|\Big(\frac{(x-s)^{2-\alpha(x)}}{2-\alpha(x)}\Big)'\Big| \\
&\quad = \Big|\frac{\alpha'(x)(x-s)^{2-\alpha(x)}\big(1-(2-\alpha(x))\ln(x-s)\big)}{(2-\alpha(x))^2} + (x-s)^{1-\alpha(x)}\Big| \\
&\quad \le \frac{Q}{(x-s)^{\alpha(x)-1}} = \frac{Q(x-s)^{\alpha^*-\alpha(x)}}{(x-s)^{\alpha^*-1}} \le \frac{Q}{(x-s)^{\alpha^*-1}}.
\end{aligned} \tag{6.9}$$

We integrate J_4 by parts, differentiate the resulting terms and use (6.9) to get

$$\begin{aligned}
|J_4| &= \frac{|k(x)|}{\Gamma(2-\alpha(x))}\Big|\Big(\frac{(x-\varepsilon)^{2-\alpha(x)}}{2-\alpha(x)}\Big)'v'(\varepsilon) + \int_\varepsilon^t v''(s)\Big(\frac{(x-s)^{2-\alpha(x)}}{2-\alpha(x)}\Big)'ds\Big| \\
&\le Q\|f\|_{C^1[0,1]}\varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(x)} + Q\int_\varepsilon^x \frac{|v''(s)|}{(x-s)^{\alpha(x)-1}}ds \\
&\le Q\|f\|_{C^1[0,1]}\varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)} + Q\int_\varepsilon^x \frac{|v''(s)|}{(x-s)^{\alpha^*-1}}ds, \quad x \in (\varepsilon, 1].
\end{aligned}$$

We similarly bound J_3 for any $x \in (\varepsilon, 1]$ by

$$\begin{aligned}
|J_3| &= \frac{|k(x)|}{\Gamma(2-\alpha(x))}\Big|\int_{\varepsilon/2}^{\varepsilon} v'(s)\Big[\frac{-\alpha'(x)\ln(x-s)}{(x-s)^{\alpha(x)-1}} + \frac{1-\alpha(x)}{(x-s)^{\alpha(x)}}\Big]ds\Big| \\
&\le Q\|f\|_{C^1[0,1]}\varepsilon^{1-\alpha(0)}\int_{\varepsilon/2}^{\varepsilon}(x-s)^{-\alpha(x)}ds \le \frac{Q\|f\|_{C^1[0,1]}}{\varepsilon^{\alpha(0)-1}(x-\varepsilon)^{\alpha(x)-1}} \\
&\le \frac{Q\|f\|_{C^1[0,1]}}{\varepsilon^{\alpha(0)-1}(x-\varepsilon)^{\alpha(\varepsilon)-1}}.
\end{aligned}$$

We incorporate the preceding estimates into (6.8) to conclude that for $x \in (\varepsilon, 1]$

$$|v''(x)| \le Q \int_\varepsilon^x \frac{|v''(s)|ds}{(x-s)^{\alpha^*-1}} + Q\|f\|_{C^2[0,1]}\Big(\varepsilon^{-\alpha(0)} + \varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)}\Big). \tag{6.10}$$

We apply Lemma 2.1 to (6.10) and use the fact that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(Q\Gamma(2-\alpha^*))^n}{\Gamma(n(2-\alpha^*))} \int_\varepsilon^x (x-s)^{n(2-\alpha^*)-1}\Big(\varepsilon^{-\alpha(0)} + \varepsilon^{1-\alpha(0)}(s-\varepsilon)^{1-\alpha(\varepsilon)}\Big)ds \\
&\quad \le \varepsilon^{-\alpha(0)} \sum_{n=1}^{\infty} \frac{(Q\Gamma(2-\alpha^*))^n}{\Gamma(n(2-\alpha^*)+1)}(x-\varepsilon)^{n(2-\alpha^*)} \\
&\qquad + \varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)} \sum_{n=1}^{\infty} \frac{(Q\Gamma(2-\alpha^*))^n}{\Gamma(n(2-\alpha^*)+2-\alpha(\varepsilon))}(x-\varepsilon)^{n(2-\alpha^*)} \\
&\quad \le \varepsilon^{-\alpha(0)} E_{2-\alpha^*,1}(Q\Gamma(2-\alpha^*)(x-\varepsilon)^{2-\alpha^*}) \\
&\qquad + \varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)} E_{2-\alpha^*,2-\alpha(\varepsilon)}(Q\Gamma(2-\alpha^*)(x-\varepsilon)^{2-\alpha^*}).
\end{aligned}$$

to conclude from (6.10) that for $x \in (\varepsilon, 1]$

$$\begin{aligned}
|v''| &\le Q\|f\|_{C^2[0,1]}\Big[\varepsilon^{-\alpha(0)} + \varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)} + \sum_{n=1}^{\infty} \frac{(Q\Gamma(2-\alpha^*))^n}{\Gamma(n(2-\alpha^*))} \\
&\qquad \times \int_\varepsilon^x (x-s)^{n(2-\alpha^*)-1}\Big(\varepsilon^{-\alpha(0)} + \varepsilon^{1-\alpha(0)}(s-\varepsilon)^{1-\alpha(\varepsilon)}\Big)ds\Big] \\
&\le Q\|f\|_{C^2[0,1]}\Big(\varepsilon^{-\alpha(0)} + \varepsilon^{1-\alpha(0)}(x-\varepsilon)^{1-\alpha(\varepsilon)}\Big).
\end{aligned} \tag{6.11}$$

Restricting $x \in [2\varepsilon, 1]$ in (6.11) yields an estimate $\|v\|_{C^2[2\varepsilon,1]} \le Q\|f\|_{C^2[0,1]}\varepsilon^{-\alpha(0)}$. Since $0 < \varepsilon \ll 1$ is arbitrarily small, we replace ε by $\varepsilon/2$ in the estimate to get

$$\|v\|_{C^2[\varepsilon,1]} \le Q\|f\|_{C^2[0,1]}\varepsilon^{-\alpha(0)}$$

which leads to (6.7). The proof of (6.6) can be carried out similarly and thus be omitted. □

6.2 Discretization and error estimate

We follow [113] to present an indirect collocation method for model (6.1). Let $x_i := (i/N)^r$ for $0 \le i \le N$ and $r \ge 1$ be a graded partition of $[0,1]$ that reduces to a uniform mesh for $r = 1$. Using mean-value theorem bounds $h_i := x_i - x_{i-1}$ by

$$h := \max_{1\le i\le N} h_i = \max_{1\le i\le N} \frac{i^r - (i-1)^r}{N^r} \le \max_{1\le i\le N} \frac{ri^{r-1}}{N^r} \le \frac{r}{N}, \quad 1 \le i \le N. \tag{6.12}$$

Let $v_h(x)$ be a piecewise-linear function with respect to the partition such that $v_h(x_i) = v_i$ for $i = 0, 1, \ldots, N$ and let I_h be the piecewise-linear interpolation operator on the partition. For any function $v(x)$ on $[0, 1]$, we define $\|v\|_{\hat{L}_\infty} := \max_{0\le i\le N} |v(x_i)|$. An indirect collocation method for (6.1) states as follows:

Step 1 Find $v_h(x)$ such that

$$v_h = \mathrm{I}_h\Big(- k\, {}_0I_x^{2-\alpha(x)} v_h - f\Big). \tag{6.13}$$

Step 2 Define an approximation $u_h(x)$ of the solution $u(x)$ to (6.1) by

$$u_h := v_h * x - x \cdot (v_h * x|_{x=1}). \tag{6.14}$$

6.2.1 A generalized discrete Gronwall's inequality

We prove a discrete Gronwall's inequality used in subsequent error estimates [113].

Lemma 6.4. *[113] Suppose the positive sequences $\{z_n\}_{n=1}^N$ and $\{y_n\}_{n=1}^N$ satisfy*

$$z_n \le M \sum_{i=1}^{n-1} \frac{z_i}{(n-i)^{1-\beta}} + y_n, \quad 1 \le n \le N,\ 0 < \beta < 1,\ M = M(N) > 0. \tag{6.15}$$

Then the sequence $\{z_n\}_{n=1}^N$ can be bounded from above by

$$z_n \le y_n + \sum_{m=1}^{n-1} \frac{(M\Gamma(\beta))^m}{\Gamma(m\beta)} \sum_{j=1}^{n-m} y_j (n-j)^{m\beta-1}, \quad 1 \le n \le N. \tag{6.16}$$

This is a generalization of the weakly singular Gronwall's inequality [7, Theorem 6.1.19], in which $\{y_n\}_{n=1}^N$ is assumed non-decreasing and $M(N) = M_* N^{-\beta}$ for some positive constant M_*. Under these additional assumptions, (6.16) reduces to the following estimate for $1 \le n \le N$ that was obtained in [7, Theorem 6.1.19]

$$\begin{aligned} z_n &\le y_n\Big(1 + \sum_{i=1}^{n-1} \frac{(M_* N^{-\beta}\Gamma(\beta))^i}{\Gamma(i\beta)} \sum_{j=1}^{n-i} (n-j)^{i\beta-1}\Big) \\ &\le y_n\Big(1 + \sum_{i=1}^{n-1} \frac{(M_* N^{-\beta}\Gamma(\beta))^i n^{i\beta}}{\Gamma(i\beta+1)}\Big) \le y_n(1 + E_{\beta,1}(M_*\Gamma(\beta))). \end{aligned} \tag{6.17}$$

Proof. Let $\mathbf{A} = (a_{ij})_{i,j=1}^N$ be a strictly lower triangular matrix with $a_{ij} := M/(i-j)^{1-\beta}$ for $1 \le j < i \le N$ and 0 elsewhere. For any $\mathbf{y} := (y_1, \cdots, y_N)^T$ and $\mathbf{z} = (z_1, \cdots, z_N)^T$, $\mathbf{y} \le \mathbf{z}$ implies $\mathbf{Ay} \le \mathbf{Az}$, where the inequality means it holds elementwise. (6.15) can be expressed in a matrix form

$$\mathbf{z} \le \mathbf{Az} + \mathbf{y}. \tag{6.18}$$

It is clear that the first m entries of $\mathbf{A}^m\mathbf{y}$ vanish for any $\mathbf{y} \in \mathbb{R}^N_{\ge 0}$ and $m \ge 1$. We prove by induction that the n-th entry $(\mathbf{A}^m\mathbf{y})_n$ of $A^m\mathbf{y}$ satisfies

$$(\mathbf{A}^m\mathbf{y})_n \le \frac{(M\Gamma(\beta))^m}{\Gamma(m\beta)} \sum_{j=1}^{n-m} y_j (n-j)^{m\beta-1}, \quad m+1 \le n \le N. \tag{6.19}$$

By definition of $\mathbf{A}$, the equality in (6.19) holds for $m = 1$. Assume that (6.19) holds for $m = \bar{m}$ for some $1 \le \bar{m} \le N-2$. Then for $m = \bar{m}+1$, we recall that $(\mathbf{A}^{\bar{m}}\mathbf{y})_i = 0$ for $1 \le i \le \bar{m}$ to obtain from (6.19) that for $\bar{m}+2 \le n \le N$

$$\begin{aligned}
(\mathbf{A}^{\bar{m}+1}\mathbf{y})_n &= M\sum_{i=1}^{n-1} \frac{(\mathbf{A}^{\bar{m}}\mathbf{y})_i}{(n-i)^{1-\beta}} = M\sum_{i=\bar{m}+1}^{n-1} \frac{(\mathbf{A}^{\bar{m}}\mathbf{y})_i}{(n-i)^{1-\beta}} \\
&\le M\sum_{i=\bar{m}+1}^{n-1} \frac{1}{(n-i)^{1-\beta}} \frac{(M\Gamma(\beta))^{\bar{m}}}{\Gamma(\bar{m}\beta)} \sum_{j=1}^{i-\bar{m}} y_j (i-j)^{\bar{m}\beta-1} \\
&= M\frac{(M\Gamma(\beta))^{\bar{m}}}{\Gamma(\bar{m}\beta)} \sum_{j=1}^{n-\bar{m}-1} y_j \sum_{i=j+\bar{m}}^{n-1} (n-i)^{\beta-1}(i-j)^{\bar{m}\beta-1}
\end{aligned}$$

$$\begin{aligned}
&\le M\frac{(M\Gamma(\beta))^{\bar{m}}}{\Gamma(\bar{m}\beta)} \sum_{j=1}^{n-1-\bar{m}} y_j \frac{\Gamma(\bar{m}\beta)\Gamma(\beta)}{\Gamma((\bar{m}+1)\beta)} (n-j)^{(\bar{m}+1)\beta-1} \\
&= \frac{(M\Gamma(\beta))^{\bar{m}+1}}{\Gamma((\bar{m}+1)\beta)} \sum_{j=1}^{n-(1+\bar{m})} y_j (n-j)^{(\bar{m}+1)\beta-1}.
\end{aligned}$$

By induction (6.19) holds for $1 \le m \le N-1$. In the second "$\le$" we used the estimate [19, Lemma 6.1] that for $0 < \kappa (= 1-\beta) < 1$ and $\gamma (= 1 - \bar{m}\beta) < 1$

$$\sum_{k=j+1}^{i-1} (i-k)^{-\kappa}(k-j)^{-\gamma} \le B(1-\kappa, 1-\gamma)(i-j)^{-\kappa-\gamma+1}.$$

We apply (6.18) recursively for $n-1$ times and recall $\mathbf{A}$ is nonnegative to obtain

$$\begin{aligned}
\mathbf{z} \le \mathbf{Az} + \mathbf{y} &\le \mathbf{A}\big(\mathbf{Az}+\mathbf{y}\big) + \mathbf{y} = \mathbf{A}^2\mathbf{z} + \sum_{m=0}^{1} \mathbf{A}^m\mathbf{y} \le \mathbf{A}^2\big(\mathbf{Az}+\mathbf{y}\big) + \sum_{m=0}^{1} \mathbf{A}^m\mathbf{y} \\
&= \mathbf{A}^3\mathbf{z} + \sum_{m=0}^{2} \mathbf{A}^m\mathbf{y} \le \ldots \le \mathbf{A}^n\mathbf{z} + \sum_{m=0}^{n-1} \mathbf{A}^m\mathbf{y}.
\end{aligned}$$

As $(A^n\mathbf{z})_n = 0$, we compare the nth entry of the preceding inequality and substitute the estimate (2.1) for $\mathbf{A}^m\mathbf{y}$ to obtain

$$z_n \le y_n + \sum_{m=1}^{n-1}(\mathbf{A}^m\mathbf{y})_n \le y_n + \sum_{m=1}^{n-1}\frac{(M\Gamma(\beta))^m}{\Gamma(m\beta)}\sum_{j=1}^{n-m} y_j(n-j)^{m\beta-1}, \quad 1 \le n \le N.$$

We thus finish the proof. □

6.2.2 Estimate of the truncation error

In this section, we estimate the truncation error R_n defined by

$$R_n := \frac{-k}{\Gamma(2-\alpha(x_n))}\int_0^{x_n}\frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}}ds. \tag{6.20}$$

Theorem 6.5. *[113] Suppose $f, \alpha \in C^2[0,1]$ and Assumption C holds.*

Case 1 $\alpha(0) = 1$ *and* $\alpha'(0) = 0$. *Then for* $r = 1$

$$\|R\|_{\hat{L}^\infty} \le Q\|f\|_{C^2[0,1]}N^{-2}. \tag{6.21}$$

Case 2 $\alpha(0) > 1$. *Then*

$$|R_n| \le Q\|f\|_{C^2[0,1]}n^{1-\alpha^*}N^{-(4-2\alpha^*)}, \quad 0 \le n \le N, \quad r = 1, \tag{6.22}$$

$$\|R\|_{\hat{L}^\infty} \le Q\|f\|_{C^2[0,1]}N^{-2}, \quad r \ge 2/(2-\alpha(0)). \tag{6.23}$$

Here $Q = Q(\alpha^*, \|\alpha\|_{C^2[0,1]}, k)$.

Proof. Let $G_i^{(1)}(y;x) := (x_i - x)/h_i$ for $y \in [x_{i-1}, x]$ or $-(x - x_{i-1})/h_i$ for $y \in [x, x_i]$ and $G_i^{(2)}(y;x) := -(x_i - x)(y - x_{i-1})/h_i$ for $y \in [x_{i-1}, x]$ or $-(x - x_{i-1})(x_i - y)/h_i$ for $y \in [x, x_i]$. The error expansions for a linear interpolation hold

$$v(x) - \mathrm{I}_h v(x)\Big|_{[x_{i-1},x_i]} = \int_{x_{i-1}}^{x_i} G_i^{(m)}(y;x)\frac{d^m v(y)}{d^m y}dy, \quad 1 \le i \le N, \ m = 1, 2. \tag{6.24}$$

For Case 1, $v \in C^2[0,1]$ by Theorem 6.3. We use (6.24) with $m = 2$ to get

$$|R_n| \le Q\|v\|_{C^2}N^{-2}\int_0^{x_n}(x_n - s)^{1-\alpha(x_n)}ds \le Q\|f\|_{C^2}N^{-2}.$$

For Case 2, we use (6.24) with $m = 1$, (6.5) and $x_n^{2-\alpha^*} - (x_n - x_1)^{2-\alpha^*} \le x_1^{2-\alpha^*}$ with $h_1 = x_1 = N^{-r}$ to bound the integral on the first interval $[0, x_1]$ in (6.20) by

$$\begin{aligned}
&\left| \int_0^{x_1} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds \right| \le \int_0^{x_1} \frac{\int_0^{x_1} |v'(y)| dy}{(x_n - s)^{\alpha(x_n)-1}} ds \\
&\le Q\|f\|_{C^1} \int_0^{x_1} \frac{\int_0^{x_1} y^{1-\alpha(0)} dy}{(x_n - s)^{\alpha(x_n)-1}} ds = Q\|f\|_{C^1} h_1^{2-\alpha(0)} \int_0^{x_1} (x_n - s)^{1-\alpha(x_n)} ds \\
&\le Q\|f\|_{C^1} h_1^{2-\alpha(0)} \Big(x_n^{2-\alpha^*} - (x_n - x_1)^{2-\alpha^*} \Big) \le Q\|f\|_{C^1} N^{-r(4-\alpha(0)-\alpha^*)}.
\end{aligned}$$

In a similar fashion we use (6.24) with $m = 2$ and (6.7) to bound the remaining element integrals on $[x_{i-1}, x_i]$ for $2 \le i \le n$ in (6.20) as we did in (6.34)

$$\begin{aligned}
&\left| \int_{x_{i-1}}^{x_i} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds \right| \le h_i \int_{x_{i-1}}^{x_i} \frac{\int_{x_{i-1}}^{x_i} |v''(y)| dy}{(x_n - s)^{\alpha^*-1}} ds \\
&\le Q\|f\|_{C^2} h_i \int_{x_{i-1}}^{x_i} \frac{\int_{x_{i-1}}^{x_i} y^{-\alpha(0)} dy}{(x_n - s)^{\alpha^*-1}} ds \le Q\|f\|_{C^2}\, h_i^2 x_{i-1}^{-\alpha(0)} \int_{x_{i-1}}^{x_i} (x_n - s)^{1-\alpha^*} ds \qquad (6.25) \\
&\le Q\|f\|_{C^2}\, h_i^2 x_{i-1}^{-\alpha(0)} \Big((x_n - x_{i-1})^{2-\alpha^*} - (x_n - x_i)^{2-\alpha^*} \Big).
\end{aligned}$$

We use (6.12) to bound the integral on the last interval $[x_{n-1}, x_n]$ in (6.25) by

$$\begin{aligned}
&\left| \int_{x_{n-1}}^{x_n} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds \right| \le Q\|f\|_{C^2} h_n^{4-\alpha^*} x_{n-1}^{-\alpha(0)} \\
&\le Q\|f\|_{C^2} \frac{n^{(4-\alpha^*)(r-1)}}{N^{(4-\alpha^*)r}} \frac{(n-1)^{-\alpha(0)r}}{N^{-\alpha(0)r}} \le Q\|f\|_{C^2} \frac{n^{r(4-\alpha(0)-\alpha^*)-(4-\alpha^*)}}{N^{r(4-\alpha(0)-\alpha^*)}}.
\end{aligned}$$

We use (6.12) and the facts that $x_i \ge 2^{-r} x_n$ for $\lceil n/2 \rceil \le i \le n$ and that h_i is increasing with respect to i to bound the integral on the interval $[x_{\lceil n/2 \rceil}, x_{n-1}]$ in (6.25) by

$$\begin{aligned}
&\left| \int_{x_{\lceil n/2 \rceil}}^{x_{n-1}} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds \right| \\
&\le Q\|f\|_{C^2} \sum_{i=\lceil n/2 \rceil+1}^{n-1} h_i^2 x_{i-1}^{-\alpha(0)} \Big((x_n - x_{i-1})^{2-\alpha^*} - (x_n - x_i)^{2-\alpha^*} \Big) \\
&\le Q\|f\|_{C^2} x_n^{-\alpha(0)} h_n^2 (x_n - x_{\lceil n/2 \rceil})^{2-\alpha^*} \le Q\|f\|_{C^2} x_n^{2-\alpha(0)-\alpha^*} h_n^2 \\
&\le Q\|f\|_{C^2} \frac{n^{r(2-\alpha(0)-\alpha^*)}}{N^{r(2-\alpha(0)-\alpha^*)}} \frac{n^{2(r-1)}}{N^{2r}} \le Q\|f\|_{C^2} \frac{n^{r(4-\alpha(0)-\alpha^*)-2}}{N^{r(4-\alpha(0)-\alpha^*)}}.
\end{aligned}$$

We use (6.12), the mean value theorem and the fact that $(x_n - x_i)^{1-\alpha^*} \leq Q x_n^{1-\alpha^*}$ for $1 \leq i \leq \lceil n/2 \rceil$ to bound the integral on the interval $[x_1, x_{\lceil n/2 \rceil}]$ in (6.25) by

$$\begin{aligned}
&\left| \int_{x_1}^{x_{\lceil n/2 \rceil}} \frac{v(s) - \mathrm{I}_h v(s)}{(x_n - s)^{\alpha(x_n)-1}} ds \right| \leq Q\|f\|_{C^2} \sum_{i=2}^{\lceil n/2 \rceil} h_i^2 x_{i-1}^{-\alpha(0)} (x_n - x_i)^{1-\alpha^*} h_i \\
&\leq Q\|f\|_{C^2} x_n^{1-\alpha^*} \sum_{i=2}^{\lceil n/2 \rceil} x_{i-1}^{-\alpha(0)} h_i^3 \leq \frac{Q\|f\|_{C^2} n^{r(1-\alpha^*)}}{N^{r(1-\alpha^*)}} \sum_{i=2}^{\lceil n/2 \rceil} \frac{(i-1)^{-r\alpha(0)}}{N^{-r\alpha(0)}} \frac{i^{3(r-1)}}{N^{3r}} \\
&= \frac{Q\|f\|_{C^2} n^{r(1-\alpha^*)}}{N^{r(4-\alpha(0)-\alpha^*)}} \sum_{i=2}^{\lceil n/2 \rceil} i^{r(2-\alpha(0))+r-3} \\
&\leq \begin{cases} \dfrac{Q\|f\|_{C^2} n^{1-\alpha^*}}{N^{4-2\alpha^*}}, \quad r = 1, \\ \dfrac{Q\|f\|_{C^2} n^{r(1-\alpha^*)}}{N^{r(4-\alpha(0)-\alpha^*)}} \displaystyle\sum_{i=2}^{\lceil n/2 \rceil} i^{r-1} \leq \dfrac{Q\|f\|_{C^2} n^{r(2-\alpha^*)}}{N^{r(2-\alpha^*)} N^2} \leq \dfrac{Q\|f\|_{C^2}}{N^2}, \quad r \geq \dfrac{2}{2-\alpha(0)}. \end{cases}
\end{aligned}$$

We collect the preceding estimates to complete the proof. □

6.2.3 Error estimates of $v - v_h$

Theorem 6.6. *[113] Suppose $f, \alpha \in C^2[0,1]$, Assumption C holds and h small enough.*

Case 1 $\alpha(0) = 1$, $\alpha'(0) = 0$. *Then an optimal error estimate holds for scheme* (6.13) *on a uniform mesh*

$$\|v - v_h\|_{\hat{L}^\infty} \leq Q N^{-2} \|f\|_{C^2[0,1]}. \tag{6.26}$$

Case 2 $\alpha(0) > 1$. *Then a suboptimal pointwise error estimate holds for scheme* (6.13) *on a uniform mesh*

$$|v(x_n) - v_h(x_n)| \leq Q\|f\|_{C^2[0,1]} n^{1-\alpha^*} N^{-(4-2\alpha^*)}, \quad 0 \leq n \leq N. \tag{6.27}$$

In addition, an optimal error estimate holds for scheme (6.13) *on a graded mesh*

$$\|v - v_h\|_{\hat{L}^\infty} \leq Q\|f\|_{C^2[0,1]} N^{-2}, \quad r \geq 2/(2-\alpha(0)). \tag{6.28}$$

Here $Q = Q(\alpha^*, \|\alpha\|_{C^2[0,1]}, k)$.

Proof. We let $e := v - v_h$ and subtract (6.13) from (6.2) to obtain an error equation

$$e(x_n) = \frac{-k}{\Gamma(2-\alpha(x_n))} \int_0^{x_n} \frac{\mathrm{I}_h e(s)}{(x_n - s)^{\alpha(x_n)-1}} ds + R_n.$$

Here the local truncation error r_n is given by (6.20). We use $e(x_0) = 0$ to bound $e(x_n)$ from the error equation as follows

$$\begin{aligned}
|e(x_n)| &\leq Q \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \frac{|\mathrm{I}_h e(s)| ds}{(x_n - s)^{\alpha^*-1}} + |R_n| \\
&\leq Q \sum_{i=1}^{n} (|e(x_{i-1})| + |e(x_i)|) \int_{x_{i-1}}^{x_i} \frac{1}{(x_n - s)^{\alpha^*-1}} ds + |R_n| \\
&\leq Q \sum_{i=1}^{n} \Big(|e(x_{i-1})| + |e(x_i)|\Big)\Big((x_n - x_{i-1})^{2-\alpha^*} - (x_n - x_i)^{2-\alpha^*}\Big) + |R_n| \\
&= Q|e(x_n)|h_n^{2-\alpha^*} + \sum_{i=1}^{n-1} |e(x_i)|\Big((x_n - x_{i-1})^{2-\alpha^*} - (x_n - x_{i+1})^{2-\alpha^*}\Big) + |R_n|.
\end{aligned} \tag{6.29}$$

We use the following elementary estimates for $1 \leq i \leq n-2$

$$\begin{aligned}
(x_n - x_{i-1})^{2-\alpha^*} - (x_n - x_{i+1})^{2-\alpha^*} &\leq (2-\alpha^*)(x_{i+1} - x_{i-1})(x_n - x_{i+1})^{1-\alpha^*}, \\
\frac{(n-i)^{\alpha^*-1}}{(n-(i+1))^{\alpha^*-1}} &= \Big(1 + \frac{1}{n-(i+1)}\Big)^{\alpha^*-1} \leq 2^{\alpha^*-1}, \\
x_n - x_{i+1} = \Big(\frac{n}{N}\Big)^r - \Big(\frac{i+1}{N}\Big)^r &\geq r\Big(\frac{i+1}{N}\Big)^{r-1}\Big(\frac{n-(i+1)}{N}\Big), \\
x_{i+1} - x_{i-1} = \Big(\frac{i+1}{N}\Big)^r - \Big(\frac{i-1}{N}\Big)^r &\leq r\Big(\frac{i+1}{N}\Big)^{r-1}\frac{2}{N}
\end{aligned}$$

and recall $1 < \alpha^* < 2$ to bound the factors $(x_n - x_{i-1})^{2-\alpha^*} - (x_n - x_{i+1})^{2-\alpha^*}$ by

$$\begin{aligned}
&(x_n - x_{i-1})^{2-\alpha^*} - (x_n - x_{i+1})^{2-\alpha^*} \\
&\quad \leq (2-\alpha^*) r\Big(\frac{i+1}{N}\Big)^{r-1}\frac{2}{N}\Big[r\Big(\frac{i+1}{N}\Big)^{r-1}\Big(\frac{n-(i+1)}{N}\Big)\Big]^{1-\alpha^*} \\
&\quad = 2(2-\alpha^*) r^{2-\alpha^*}\Big(\frac{i+1}{N}\Big)^{(r-1)(2-\alpha^*)} \frac{1}{N^{2-\alpha^*}} \frac{1}{(n-(i+1))^{\alpha^*-1}} \\
&\quad \leq \frac{Q}{N^{2-\alpha^*}(n-(i+1))^{\alpha^*-1}} \leq \frac{Q}{N^{2-\alpha^*}(n-i)^{\alpha^*-1}}.
\end{aligned}$$

We choose $Qh^{2-\alpha^*} \leq 1/2$ to cancel $|e(x_n)|$ on both sides of (6.29) to obtain

$$|e(x_n)| \leq M_1 N^{-(2-\alpha^*)} \sum_{i=1}^{n-1} \frac{|e(x_i)|}{(n-i)^{\alpha^*-1}} + M_2 |R_n|, \quad 1 \leq n \leq N. \tag{6.30}$$

For Case 1, $r = 1$. We incorporate the upper bound (6.21) for r_n into (6.30) and apply the generalized Gronwall's inequality (6.17) with $\beta = 2 - \alpha^*$ to prove (6.26)

$$|e(x_n)| \leq M_2 |r_n| (1 + E_{2-\alpha^*,1}(M_1 \Gamma(2-\alpha^*))) \leq Q\|f\|_{C^2[0,1]} N^{-2}.$$

For Case 2 with a graded mesh of $r \geq 2/(2-\alpha(0))$, we similarly prove estimate (6.28) by incorporating the upper bound (6.23) for r_n into (6.30) and apply Gronwall's inequality (6.17) with $\beta = 2-\alpha^*$. To prove estimate (6.27), we incorporate the upper bound (6.22) for r_n into (6.30) and apply Lemma 6.4 with $\beta = 2-\alpha^*$ to obtain

$$
\begin{aligned}
|e(x_n)| &\leq M_2|r_n| + M_2 \sum_{i=1}^{n-1} \frac{(M_1 N^{-(2-\alpha^*)}\Gamma(2-\alpha^*))^i}{\Gamma(i(2-\alpha^*))} \sum_{j=1}^{n-i} |r_j|(n-j)^{i(2-\alpha^*)-1} \\
&\leq QM_2\|f\|_{C^2}\Bigg[n^{1-\alpha^*}N^{-(4-2\alpha_M)} \\
&\quad + \sum_{i=1}^{n-1} \frac{(M_1 N^{-(2-\alpha^*)}\Gamma(2-\alpha^*))^i}{\Gamma(i(2-\alpha^*))} \sum_{j=1}^{n-i} \frac{j^{1-\alpha^*}}{N^{4-2\alpha^*}}(n-j)^{i(2-\alpha^*)-1}\Bigg] \\
&\leq QM_2\|f\|_{C^2}\Bigg[n^{1-\alpha^*}N^{-(4-2\alpha^*)} \\
&\quad + \frac{1}{N^{4-2\alpha^*}} \sum_{i=1}^{n-1} \frac{(M_1 N^{-(2-\alpha^*)}\Gamma(2-\alpha^*))^i}{\Gamma(i(2-\alpha^*))} \int_0^n x^{1-\alpha^*}(n-x)^{i(2-\alpha^*)-1}dx\Bigg] \\
&= QM_2\|f\|_{C^2}\Bigg[n^{1-\alpha^*}N^{-(4-2\alpha^*)} \\
&\quad + \frac{1}{N^{4-2\alpha^*}} \sum_{i=1}^{n-1} \frac{(M_1 N^{-(2-\alpha^*)}\Gamma(2-\alpha^*))^i}{\Gamma((i+1)(2-\alpha^*))}\Gamma(2-\alpha^*)n^{(i+1)(2-\alpha^*)-1}\Bigg] \\
&= \frac{QM_2\|f\|_{C^2}}{N^{4-2\alpha^*}n^{\alpha^*-1}}\left[1+\Gamma(2-\alpha^*)\sum_{i=1}^{n-1}\frac{(M_1 N^{-(2-\alpha^*)}\Gamma(2-\alpha^*)n^{2-\alpha^*})^i}{\Gamma((i+1)(2-\alpha^*))}\right] \\
&\leq Q\|f\|_{C^2}n^{1-\alpha^*}N^{-(4-2\alpha^*)}\left(1+\Gamma(2-\alpha^*)\sum_{i=1}^{n-1}\frac{(M_1\Gamma(2-\alpha^*))^i}{\Gamma((i+1)(2-\alpha^*))}\right) \\
&\leq Q\|f\|_{C^2}n^{1-\alpha^*}N^{-(4-2\alpha^*)}\Big(1+\Gamma(2-\alpha^*)E_{2-\alpha^*,2-\alpha^*}(M_1\Gamma(2-\alpha^*))\Big) \\
&\leq Q\|f\|_{C^2}n^{1-\alpha^*}N^{-(4-2\alpha^*)}.
\end{aligned}
$$

We finish the proof of the theorem. □

6.2.4 Error estimates of $u - u_h$

Theorem 6.7. *[113] Suppose $f, \alpha \in C^2[0,1]$, Assumption C holds and h small enough.*

Case 1 *$\alpha(0) = 1$ and $\alpha'(0) = 0$. Then an optimal error estimate holds for scheme (6.14) on a uniform mesh*

$$\|u-u_h\|_{L_\infty} \leq Q\|f\|_{C^2[0,1]}N^{-2}. \tag{6.31}$$

Case 2 $\alpha(0) > 1$. *Then suboptimal and optimal error estimates hold on a uniform mesh and a graded mesh, respectively*

$$\|u - u_h\|_{L_\infty} \leq Q\|f\|_{C^2[0,1]} N^{-(3-\alpha^*)}, \qquad r = 1,$$

$$\|u - u_h\|_{L_\infty} \leq Q\|f\|_{C^2[0,1]} N^{-2}, \qquad r \geq 2/(2 - \alpha(0)).$$

Here $Q = Q(\alpha^*, \|\alpha\|_{C^2[0,1]}, k)$.

Proof. We subtract (6.3) from (6.14) to obtain

$$\begin{aligned} |u(x) - u_h(x)| &= \left| \int_0^x (v(s) - v_h(s))(x - s)ds - x \int_0^1 (v(s) - v_h(s))(1 - s)ds \right| \\ &\leq 2 \int_0^1 |v(s) - \mathrm{I}_h v(s)|ds + 2 \int_0^1 |\mathrm{I}_h(v(s) - v_h(s))|ds \\ &\leq 2 \int_0^1 |v(s) - \mathrm{I}_h v(s)|ds + Q \sum_{i=1}^N h_i \Big| v(x_i) - v_h(x_i) \Big|. \end{aligned} \tag{6.32}$$

In Case 1, $v \in C^2[0,1]$. We incorporate (6.21) and (6.24) with $m = 2$ into (6.32) to obtain (6.31). For Case 2, we estimate the first term on the right-hand side of (6.32) elementwise. We use estimate (6.5) and expansion (6.24) with $m = 1$ to get the following estimate

$$\begin{aligned} \int_0^{x_1} |v(s) - \mathrm{I}_h v(s)|ds &\leq \int_0^{x_1} \int_0^{x_1} |v'(y)|dyds \\ &\leq Q h_1 \|f\|_{C^1[0,1]} \int_0^{x_1} y^{1-\alpha(0)} dy \\ &\leq Q\|f\|_{C^1[0,1]} h_1^{3-\alpha(0)}. \end{aligned} \tag{6.33}$$

We use estimate (6.7) and expansion (6.24) with $m = 2$ to bound the integral on $[x_{i-1}, x_i]$ by

$$\begin{aligned} \int_{x_{i-1}}^{x_i} |v(s) - \mathrm{I}_h v(s)|ds &\leq h_i \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} |v''(y)|dyds \\ &\leq Q\|f\|_{C^2[0,1]} h_i^2 \int_{x_{i-1}}^{x_i} y^{-\alpha(0)} dy \\ &\leq Q\|f\|_{C^2[0,1]}\, h_i^3 x_{i-1}^{-\alpha(0)}. \end{aligned} \tag{6.34}$$

We combine (6.33) and (6.34) with (6.12) to bound the first term on the right-hand side of (6.32) by

$$\begin{aligned}
\int_0^1 |v(s) - \mathrm{I}_h v(s)| ds &\le Q\|f\|_{C^2[0,1]} \Big(N^{-r(3-\alpha(0))} + \sum_{i=2}^N h_i^3 x_{i-1}^{-\alpha(0)} \Big) \\
&\le Q\|f\|_{C^2} \Big(N^{-r(3-\alpha(0))} + \sum_{i=2}^N i^{3(r-1)} N^{-3r} (i-1)^{-r\alpha(0)} N^{r\alpha(0)} \Big) \\
&\le Q\|f\|_{C^2} N^{-r(3-\alpha(0))} \Big(1 + \sum_{i=2}^N i^{r(2-\alpha(0))+r-3} \Big) \\
&\le \begin{cases} Q\|f\|_{C^2} N^{-(3-\alpha(0))} \sum_{i=1}^N i^{-\alpha(0)} \le Q\|f\|_{C^2} N^{-(3-\alpha(0))}, & r = 1, \\ Q\|f\|_{C^2} N^{-r(3-\alpha(0))} \sum_{i=1}^N i^{r-1} \le Q\|f\|_{C^2} N^{-2}, & r \ge \dfrac{2}{2-\alpha(0)}, \end{cases}
\end{aligned} \tag{6.35}$$

where we have used the facts that $\sum_{i=1}^N i^{-\alpha(0)} < \infty$ and $\sum_{i=1}^N i^{r-1} \le N^r$.

For $r = 1$ we use (6.27), (6.32) and (6.35) to bound $\|u - u_h\|_{L^\infty}$ by

$$\begin{aligned}
\|u - u_h\|_{L^\infty} &\le Q\|f\|_{C^2} \Big(N^{-(3-\alpha^*)} + \sum_{i=1}^N h_i i^{1-\alpha^*} N^{-(4-2\alpha^*)} \Big) \\
&= Q\|f\|_{C^2} \Big(N^{-(3-\alpha^*)} + \sum_{i=1}^N N^{-1} i^{1-\alpha^*} N^{-(4-2\alpha^*)} \Big) \\
&\le Q\|f\|_{C^2} \big(N^{-(3-\alpha^*)} + N^{2-\alpha^*} N^{-(5-2\alpha^*)} \big) \le Q\|f\|_{C^2} N^{-(3-\alpha^*)}.
\end{aligned}$$

We combine (6.28), (6.32) and (6.35) to bound $u - u_h$ for $r \ge 2/(2-\alpha(0))$

$$\|u - u_h\|_{L^\infty} \le Q\|f\|_{C^2} \Big(N^{-2} + N^{-2} \sum_{i=1}^N h_i \Big) \le Q\|f\|_{C^2} N^{-2}.$$

We thus finish the entire proof. □

6.3 Numerical experiments

We follow [113] to numerically observe the impact of $\alpha(0)$ on the regularity of the solutions and the convergence rates of the numerical approximations. We choose $k = 1$ and $f(x) = 1$ on $x \in [0, 1]$ and a variable order of the form

$$\alpha(x) = \alpha(1) + (\alpha(0) - \alpha(1))\big((1-x) - \sin(2\pi(1-x))/(2\pi)\big). \tag{6.36}$$

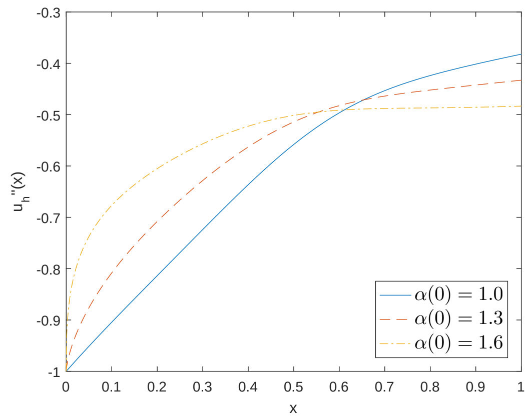

Figure 6.1 Plots of $\partial_{xx}u_h(x)$ on $x \in [0,1]$ for cases (i)-(iii) [113, FIG. 1].

6.3.1 Singular behavior of solutions at $x = 0^+$

We present the plots of $\partial_{xx}u_h(x)$ for three scenarios: (i) $\alpha(0) = 1.0$ and $\alpha(1) = 1.3$; (ii) $\alpha(0) = 1.3$ and $\alpha(1) = 1.6$; and (iii) $\alpha(0) = 1.6$ and $\alpha(1) = 1.9$ with a uniform mesh size $h = 1/1000$ in Figure 6.1. We observe that for $\alpha(0) \neq 1$, the solutions exhibit singular behavior near $x = 0$ that gets stronger as $\alpha(0)$ increases, which is consistent with Theorems 6.2 and 6.3.

6.3.2 Convergence rates of v_h and u_h

In Tables 6.1–6.3 we present the errors $u - u_h$ and $v - v_h$ and the convergence rates for different $\alpha(0)$ and $\alpha(1)$ in (6.36)

$$\|u - u_h\|_{L^\infty} \leq QN^{-\mu}, \quad \|v - v_h\|_{\hat{L}^\infty} \leq QN^{-\nu}$$

Since the true solution is not available, we compute the reference solutions using a fine mesh size of $N = 2880$ on a uniform (denoted by 'U') or graded (denoted by 'G') mesh.

We observe that for $\alpha(0) > 1$, the numerical approximations u_h and v_h have suboptimal order convergence rates of $\mu = 3 - \alpha_M$ and $\nu = 4 - 2\alpha_M$, respectively, on a uniform mesh, and optimal-order convergence rates on a graded mesh of $r =$

$2/(2-\alpha(0))$. Moreover, u_h and v_h have optimal-order convergence rates as long as $\alpha(0) = 1$. These observations coincide with the conclusions of Theorems 6.6 and 6.7.

Table 6.1 Convergence rates for $\alpha(0) = 1.3$ and $\alpha(1) = 1.1$ [113, TABLE 1]

N	U	μ	G	μ	U	ν	G	ν
48	1.00E-05		2.75E-06		4.09E-04		3.43E-05	
72	5.29E-06	1.57	1.22E-06	2.00	2.31E-04	1.41	1.52E-05	2.00
96	3.34E-06	1.59	6.87E-07	2.00	1.54E-04	1.41	8.57E-06	2.00
120	2.34E-06	1.61	4.39E-07	2.01	1.12E-04	1.42	5.47E-06	2.01
144	1.74E-06	1.62	3.05E-07	2.01	8.62E-05	1.43	3.79E-06	2.02

Table 6.2 Convergence rates for $\alpha(0) = 1.6$ and $\alpha(1) = 1.4$ [113, TABLE 2]

N	U	μ	G	μ	U	ν	G	ν
48	4.53E-05		3.57E-06		5.99E-03		6.09E-05	
72	2.65E-05	1.33	1.57E-06	2.02	4.45E-03	0.73	2.75E-05	1.96
96	1.80E-05	1.35	8.83E-07	2.01	3.59E-03	0.75	1.56E-05	1.97
120	1.32E-05	1.36	5.65E-07	2.00	3.03E-03	0.76	1.01E-05	1.98
144	1.03E-05	1.38	3.93E-07	1.99	2.63E-03	0.78	7.00E-06	1.98

Table 6.3 Convergence rates for $\alpha(0) = 1.0$ and $\alpha(1) = 1.2$ or 1.8 [113, TABLE 3]

$\alpha(1)$	1.2		1.8		1.2		1.8	
N	U	μ	U	μ	U	ν	U	ν
48	2.72E-06		5.94E-06		1.93E-05		5.57E-05	
72	1.21E-06	2.00	2.63E-06	2.01	8.57E-06	2.00	2.50E-05	1.98
96	6.80E-07	2.00	1.48E-06	2.01	4.81E-06	2.01	1.41E-05	1.98
120	4.35E-07	2.00	9.44E-07	2.01	3.07E-06	2.01	9.07E-06	1.99
144	3.02E-07	2.00	6.54E-07	2.01	2.13E-06	2.02	6.30E-06	1.99

Chapter 7

Inverse variable order problems of time/space fractional PDEs

In real applications, the (variable) fractional orders in fractional PDEs are usually unknown and need to be determined and inferred from the observations of, e.g., solutions. In particular, the uniqueness of the identification is the key to guarantee the reliability of the experimentally inferred parameters. There are extensive investigations on the inverse fractional order problems of fractional PDEs (see e.g., [33, 35, 51, 54, 59]), while the corresponding studies on variable-order fractional models are still not available in the literature to our best knowledge. In this chapter we study the unique determination of the variable fractional orders in the variable-order tFDE (3.1) and the variable-order sFDE (6.1), with the observations of the unknown solutions on any arbitrarily small spatial domain (over a sufficiently small time interval). The proved theorems provide guidance where the measurements should be performed, and ensure that with these observations the uniqueness of the identification is theoretically guaranteed.

We first study the unique determination of the variable fractional order in the variable-order tFDE (3.1) following [123].

Lemma 7.1. *Suppose that the initial data $u_0 \in \check{H}^{2+r}$ and $f \in H^{\kappa}(\check{H}^r)$ for $r > d/2$ in problem (3.1) satisfy $\mathcal{B}u_0 - f(\boldsymbol{x}, 0) \not\equiv 0$. Then there exists an open spatial domain $\Lambda \subset \Omega$ and a positive constant $\sigma > 0$ such that either $\mathcal{B}u_0 - f(\boldsymbol{x}, 0) \geq \sigma$ or $\mathcal{B}u_0 - f(\boldsymbol{x}, 0) \leq -\sigma$ for $\boldsymbol{x} \in \Lambda$.*

Proof. By the Sobolev embedding $H^{2+d/2+\varepsilon}(\Omega) \hookrightarrow C^2(\Omega)$ for any $\varepsilon > 0$ and the fact that $\check{H}^s(\Omega)$ is a subspace of $H^s(\Omega)$, $u_0 \in \check{H}^{2+r}(\Omega)$ implies $u_0 \in C^2(\Omega)$ and $f \in H^\kappa(\check{H}^r)$ implies $f \in C([0,T]; C(\Omega))$. Then the conclusion of this lemma is a straightforward consequence of the assumption $\mathcal{B}u_0 - f(\boldsymbol{x}, 0) \not\equiv 0$. In practice, as the u_0 and f are given data, such Λ can be identified by directly calculating $\mathcal{B}u_0 - f(\boldsymbol{x}, 0)$. □

Based on this lemma, we present the main theorem of this chapter [123].

Theorem 7.2. *Suppose that* $k \neq 0$, $u_0 \in \check{H}^{r+2}$ *and* $f \in H^\kappa(\check{H}^r)$ *for* $r > 2 + d/2$ *and* $\kappa > 1/2$. *Let* $\Lambda \subset \Omega$ *be an open subset on which either* $\mathcal{B}u_0 - f(\boldsymbol{x}, 0) \geq \sigma$ *or* $\mathcal{B}u_0 - f(\boldsymbol{x}, 0) \leq -\sigma$ *holds for some constant* $\sigma > 0$. *Then the variable order* $\alpha(t)$ *in model (3.1) can be determined uniquely in the following admissible set*

$$\mathcal{A} := \Big\{\alpha(t) : \alpha(t) \text{ is analytic on } [0,T] \text{ and satisfies the Assumption } A\Big\},$$

given the observations of the solution u *to (3.1) in* Λ *over a small time interval.*

More precisely, let $\hat{u}(\boldsymbol{x}, t)$ *be the solution to the following equation*

$$\begin{aligned} &\partial_t \hat{u} + k\, \partial_t^{\hat{\alpha}(t)} \hat{u} + \mathcal{B}u = f, \quad (\boldsymbol{x}, t) \in \Omega \times (0, T]; \\ &\hat{u}(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \quad \hat{u}(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times [0, T] \end{aligned} \tag{7.1}$$

with the variable order $\hat{\alpha}(t) \in \mathcal{A}$. *If*

$$u(\boldsymbol{x}, t) = \hat{u}(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Lambda \times [0, \tau]$$

for some sufficiently small $\tau > 0$, *then we have*

$$\alpha(t) = \hat{\alpha}(t), \quad t \in [0, T].$$

Proof. By Theorem 3.2, $u_0 \in \check{H}^{2+r}(\Omega)$ and $f \in H^\kappa(\check{H}^r)$ implies $u \in C^1([0,T]; \check{H}^r(\Omega))$. As $r > 2 + d/2$, we apply the Sobolev embedding theorem [3] to conclude that $u \in C^1([0,T]; C^2(\Omega))$. By Lemma 7.1, there exists an open subset $\Lambda \subset \Omega$ such that either $\mathcal{B}u_0 - f(\boldsymbol{x}, 0) \geq \sigma$ or $\mathcal{B}u_0 - f(\boldsymbol{x}, 0) \leq -\sigma$ for $\boldsymbol{x} \in \Lambda$. In the rest of the proof we assume $\mathcal{B}u_0 - f(\boldsymbol{x}, 0) \geq \sigma$ without loss of generality.

By the continuities of $\mathcal{B}u$ and f, there exists a time interval $[0,t_0] \subset [0,T]$ such that $\mathcal{B}u - f \geq 3\sigma/4 > 0$ for $(\boldsymbol{x},t) \in \Lambda \times [0,t_0]$. We apply $\partial_t u \in C([0,T];C(\Omega))$ to conclude that $\lim_{t\to 0^+} k\,\partial_t^{\alpha(t)}u(\boldsymbol{x},t) = 0$. Then there exists a positive time instant τ with $0 < \tau \leq t_0$ such that $\left|k\,\partial_t^{\alpha(t)}u(\boldsymbol{x},t)\right| \leq \sigma/4$ for $(\boldsymbol{x},t) \in \Lambda \times [0,\tau]$. We incorporate the preceding estimates into the variable-order time-fractional partial differential equation (3.1) to conclude that

$$\partial_t u(\boldsymbol{x},t) \leq -\sigma/2 < 0, \quad \forall\, (\boldsymbol{x},t) \in \Lambda \times [0,\tau]. \tag{7.2}$$

Since $u(\boldsymbol{x},t) = \hat{u}(\boldsymbol{x},t)$ on $(\boldsymbol{x},t) \in \Lambda \times [0,\tau]$ and $u,\hat{u} \in C^1([0,T];C^2(\Omega))$, we have $\mathcal{B}u = \mathcal{B}\hat{u}$ for all $(\boldsymbol{x},t) \in \Lambda \times [0,\tau]$. We subtract equation (3.1) from equation (7.1) for $(\boldsymbol{x},t) \in \Lambda \times [0,\tau]$ to get

$$\left(\partial_t^{\alpha(t)} - \partial_t^{\hat{\alpha}(t)}\right)u(\boldsymbol{x},t) = 0, \qquad \forall (\boldsymbol{x},t) \in \Lambda \times (0,\tau].$$

An application of the mean-value theorem yields

$$\begin{aligned} 0 = \left(\partial_t^{\alpha(t)} - \partial_t^{\hat{\alpha}(t)}\right)u(\boldsymbol{x},t) &= \int_0^t \int_{\hat{\alpha}(t)}^{\alpha(t)} \partial_z\left(\frac{1}{\Gamma(1-z)(t-s)^z}\right)dz\,\partial_s u(\boldsymbol{x},s)ds \\ &= \int_0^t \frac{(t-s)^{-\bar{\alpha}(s,t)}}{\Gamma(1-\bar{\alpha}(s,t))}\left(\frac{\Gamma'(1-\bar{\alpha}(s,t))}{\Gamma(1-\bar{\alpha}(s,t))} - \ln(t-s)\right)\partial_s u(\boldsymbol{x},s)ds\,(\alpha(t)-\hat{\alpha}(t)), \end{aligned}$$

for $(\boldsymbol{x},t) \in \Lambda \times (0,\tau]$ where $\bar{\alpha}(s,t)$ lies between $\alpha(t)$ and $\hat{\alpha}(t)$ for any $0 < t \leq \tau$ and $0 < s < t$. By assumptions on $\alpha(t)$ (and $\hat{\alpha}(t)$), $\bar{\alpha}(s,t)$ lies between $\alpha(t)$ and $\hat{\alpha}(t)$ that are bounded between 0 and $\alpha_* < 1$. Consequently, there exist positive constants Q_0 and $\tau_1 \leq \tau$ such that

$$\frac{\Gamma'(1-\bar{\alpha}(s,t))}{\Gamma(1-\bar{\alpha}(s,t))} - \ln(t-s) > Q_0 > 0, \qquad 0 < s < t, \quad t \in (0,\tau_1]. \tag{7.3}$$

We incorporate the estimates (7.2)-(7.3) to conclude that

$$0 \geq \sigma Q_0 \int_0^t \frac{(t-s)^{-\bar{\alpha}(s,t)}}{2\Gamma(1-\bar{\alpha}(s,t))}ds\,|\alpha(t)-\hat{\alpha}(t)|, \quad 0 < s < t, \quad (\boldsymbol{x},t) \in \Lambda \times (0,\tau_1],$$

which implies $\alpha(t) = \hat{\alpha}(t)$ on $t \in (0,\tau_1]$. Then the proof is completed by the condition $\alpha,\hat{\alpha} \in \mathcal{A}$. □

Based on the above ideas and techniques as well as the well-posedness theorem of the variable-order sFDE (6.1) in Chapter 6, the variable fractional order in this sFDE can be unique identified in the following admissible set

$$\mathcal{F} := \Big\{\alpha(x) : \alpha(x) \text{ is analytic on } [0,1] \text{ and satisfies the Assumption C}\Big\}.$$

Theorem 7.3. *[122] Suppose $f \in C[0,1]$ with $f(0) \neq 0$, $k \neq 0$ and the Assumption C holds. Then the variable order $\alpha \in \mathcal{F}$ in model (6.1) can be uniquely determined from the observations of the solution $u(x)$ on an arbitrarily small interval near the left end point of the interval. More precisely, let $\hat{\alpha} \in \mathcal{F}$ and $\hat{u}(x)$ be the solution to the problem*

$$-\partial_{xx}\hat{u}(x) - k\,\partial_x^{\hat{\alpha}(x)}\hat{u}(x) = f(x), \quad x \in (0,1); \quad \hat{u}(0) = \hat{u}(1) = 0.$$

If there exists an ε_0 with $0 < \varepsilon_0 \ll 1$ such that

$$u(x) = \hat{u}(x), \quad \forall x \in [0, \varepsilon_0],$$

then the following equation holds

$$\alpha(x) = \hat{\alpha}(x), \quad \forall x \in [0,1].$$

Bibliography

[1] G. Acosta, J. Borthagaray, A fractional Laplace equation: regularity of solutions and finite element approximations. *SIAM J. Numer. Anal.* 55 (2017), 472–495.

[2] G. Acosta, J. Borthagaray, N. Heuer, Finite element approximations of the nonhomogeneous fractional Dirichlet problem. *IMA J. Numer. Anal.* 39 (2019), 1471–1501.

[3] R. Adams and J. Fournier, *Sobolev Spaces*, Elsevier, San Diego, 2003.

[4] A. Alikhanov, Numerical methods of solutions of boundary value problems for the multi-term variable-distributed order diffusion equation. *Appl. Math. Comput.* 268 (2015), 12–22.

[5] G. Akrivis, B. Li, C. Lubich, Combining maximal regularity and energy estimates for time discretizations of quasilinear parabolic equations. *Math. Comp.* 86 (2017), 1527–1552.

[6] P. Baveye, P. Vandevivere, B. Hoyle, P. DeLeo, D. de Lozada, Environmental impact and mechanisms of the biological clogging of saturated soils and aquifer materials. *Critical Reviews in Environmental Science and Technology* 28 (2006), 123–191.

[7] H. Brunner, *Collocation methods for Volterra integral and related functional differential equations.* Cambridge University Press, 2004.

[8] C. Canuto, M. Hussaini, A. Quarteroni, T.A. Zang, *Spectral Methods.* Springer-Verlag, Germany, 2006.

[9] C. Chen, H. Liu, X. Zheng, H. Wang, A two-grid MMOC finite element method for nonlinear variable-order time-fractional mobile/immobile advection-diffusion equations. *Comp. & Math. Appl.* 79 (2020), 2771–2783.

[10] H. Chen, M. Stynes, Blow-up of error estimates in time-fractional initial-boundary value problems. *IMA J. Numer. Anal.* (2020) to appear.

[11] S. Chen, J. Shen, Z. Zhang, Z. Zhou, A spectrally accurate approximation to subdiffusion equations using the log orthogonal functions. *SIAM J. Sci. Comput.* (2020) to appear.

[12] E. Cuesta, R. Ponce, Well-posedness, regularity, and asymptotic behavior of the continuous and discrete solutions of linear fractional integro-differential equations with order varying in time. *Electr. J. Diff. Eq.* 173 (2018), 1–27.

[13] P. Dai, Q. Wu, H. Wang, X. Zheng, An efficient matrix splitting preconditioning technique for two-dimensional unsteady space-fractional diffusion equations. *J. Comput. Appl. Math.* 371 (2020), 112673.

[14] D. del-Castillo-Negrete, Front propagation in reaction-diffusion systems with anomalous diffusion. *Boletín de la Sociedad Matemática Mexicana* 20 (2014), 87–105.

[15] W. Deng, B. Li, Z. Qian, H. Wang, Time discretization of a tempered fractional Feynman–Kac equation with measure data. *SIAM J. Numer. Anal.* 56 (2018), 3249–3275.

[16] W. Deng, B. Li, W. Tian, P. Zhang, Boundary problems for the fractional and tempered fractional operators. *SIAM J. Multiscale Model. Simul.* 16 (2018), 125–149.

[17] K. Diethelm, *The Analysis of Fractional Differential Equations*, Ser. Lecture Notes in Mathematics, Vol. 2004, Springer-Verlag, Berlin, 2010.

[18] K. Diethelm and N. Ford. A Note on the Well-Posedness of Terminal Value Problems for Fractional Differential Equations. *J. Integral Eq. Appl.* 30 (2018), 371–376.

[19] J. Dixon, S. McKee, Weakly singular discrete Gronwall inequalities. *ZAMM J. Appl. Math. Mech.* 66 (1986), 535–544.

[20] R. Du, A. Alikhanov and Z. Sun, Temporal second order difference schemes for the multi-dimensional variable-order time fractional sub-diffusion equations. *Comput & Math Appl* (2020) https://doi.org/10.1016/j.camwa.2020.01.003.

[21] V. Ervin and J. Roop, Variational formulation for the stationary fractional advection dispersion equation, *Numer. Methods. Partial Differential Eq.*, 22 (2006), 558–576.

[22] V. Ervin, N. Heuer, J. Roop, Numerical approximation of a time dependent, non-linear, fractional order diffusion equation. *SIAM J. Numer. Anal.* 45 (2007) 572-591.

[23] V. Ervin, N. Heuer and J. Roop, Regularity of the solution to 1-D fractional order diffusion equations, *Math. Comp.*, 87 (2018), 2273–2294.

[24] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, V 19, American Mathematical Society, Rhode Island, 1998.

[25] Z. Fang, H. Sun, H. Wang, A fast method for variable-order Caputo fractional derivative with applications to time-fractional diffusion equations. *Comput. & Math. Appl.* 80 (2020), 1443–1458.

[26] N. Ford, Y. Yan, An approach to construct higher order time discretisation schemes for time fractional partial differential equations with nonsmooth data. *Fract. Calc. Appl. Anal.* 20 (2017), 1076–1105.

[27] H. Fu, Y. Sun, H. Wang, X. Zheng, Stability and convergence of a Crank-Nicolson finite volume method for space fractional diffusion equations. *Appl. Numer. Math.* 139 (2019), 38–51.

[28] Z. Fu, S. Reutskiy, H. Sun, J. Ma, M. Khan, A robust kernel-based solver for variable-order time fractional PDEs under 2D/3D irregular domains. *Appl. Math. Lett.* 94 (2019), 105–111.

[29] L. Gandossi and U. Von Estorff, An overview of hydraulic fracturing and other formation stimulation technologies for shale gas production, Scientific and Technical Research Reports, Joint Research Centre of the European Commission; Publications Office of the European Union, 2015. doi:10.2790/379646.

[30] R. Garrappa, I. Moret, M. Popolizio, Solving the time-fractional Schrödinger equation by Krylov projection methods. *J. Comput. Phys.* 293 (2015), 115–134.

[31] R. Gorenflo. Y. Luchko, M. Yamamoto. Time-fractional diffusion equation in the fractional Sobolev spaces. *Fract. Calc. Appl. Anal.* 18 (2015), 799–820.

[32] X. Guo, Y. Li, and H. Wang, A high order finite difference method for tempered fractional diffusion equations with applications to the CGMY model. *SIAM J. Sci. Comput.* 40 (2018), A3322–A3343.

[33] Y. Hatano, J. Nakagawa, S. Wang and M. Yamamoto, Determination of order in fractional diffusion equation, *J. Math-for-Ind.* 5A (2013), 51–57.

[34] N. Heuer, On the equivalence of fractional-order Sobolev semi-norms. J. Math. Anal. Appl. 417 (2014), 505–518.

[35] J. Janno, Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time fractional diffusion equation, *Elect J. Differential Eqs.* 199 (2016), 1–28.

[36] C. Ji, W. Dai, Z. Sun, Numerical schemes for solving the time-fractional dual-phase-lagging heat conduction model in a double-layered nanoscale thin film. *J Sci. Comput.* 81 (2019), 1767–1800.

[37] J. Jia, X. Zheng, H. Fu, P. Dai and H. Wang, A fast method for variable-order space-fractional diffusion equations. *Numer. Algor.* (2020) https://doi.org/10.1007/s11075-020-00875-z.

[38] B. Jin, B. Li, Z. Zhou, Correction of high-order BDF convolution quadrature for fractional evolution equations. *SIAM J. Sci. Comput.* 39 (2017) A3129–A3152.

[39] B. Jin, R. Lazarov, J. Pasciak, W. Rundell, Variational formulation of problems involving fractional order differential operators. *Math. Comp.*, 84 (2015), 2665–2700.

[40] B. Jin, B. Li, and Z. Zhou. Numerical analysis of nonlinear subdiffusion equations. *SIAM J. Numer. Anal.* 56 (2018), :1–23.

[41] Y. Kian, E. Soccorsi, M. Yamamoto, On time-fractional diffusion equations with space-dependent variable order. *Annales Henri Poincare* 19 (2018), 3855–3881.

[42] N. Kopteva, Error analysis of the L1 method on graded and uniform meshes for a fractional-derivative problem in two and three dimensions. *Math. Comp.* 88 (2019), 2135–2155.

[43] N. Kopteva, X. Meng, Error analysis for a fractional-derivative parabolic problem on quasi-graded meshes using barrier functions. *SIAM J. Numer. Anal.* 58 (2020), 1217–1238.

[44] A. Kilbas, H. Srivastava and J. Trujillo, *Theory and applications of fractional differential equations*, 204. Elsevier B.V., 2006.

[45] M. Le, W. McLean, K. Mustapha, Numerical solution of the time-fractional Fokker-Planck equation with general forcing, *SIAM J. Numer. Anal.*, 54 (2016), 1763–1784.

[46] K. Le, M. Stynes, An α-robust semidiscrete finite element method for a Fokker-Planck initial-boundary value problem with variable-order fractional time derivative. DOI: 10.13140/RG.2.2.27575.14249.

[47] B. Li, H. Luo, X. Xie, Analysis of a time-stepping scheme for time fractional diffusion problems with nonsmooth data. *SIAM J. Numer. Anal.* 57 (2019), 779–798.

[48] C. Li, Q. Yi, Finite difference method for two-dimensional nonlinear time-fractional subdiffusion equation. *Fract. Calc. Appl. Anal.* 21 (2018), 1046–1072.

[49] C. Li, Q. Yi, A. Chen, Finite difference methods with non-uniform meshes for nonlinear fractional differential equations. *J. Comput. Phys.* 316 (2016), 614–631.

[50] D. Li, J. Wang, J. Zhang, Unconditionally convergent $L1$-Galerkin FEMs for nonlinear time-fractional Schrödinger equations. *SIAM J. Sci. Comput.* 39 (2017), A3067–A3088.

[51] G. Li, D. Zhang, X. Jia and M. Yamamoto, Simultaneous inversion for the space-dependent diffusion coefficient and the fractional order in the time-fractional diffusion equation, *Inverse Problems* 29 (2013), 065014.

[52] X. Li, C. Xu, A space-time spectral method for the time fractional diffusion equation. *SIAM J. Numer. Anal.* 47 (2009), 2108–2131.

[53] Z. Li, H. Wang, R. Xiao and S. Yang, A variable order fractional differential equation model of shape memory polymers, *Chaos, Solitons & Fractals*, 102 (2017), 473–485.

[54] Z. Li and M. Yamamoto, Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation. *Appl Anal* 94 (2015), 570–579.

[55] H. Liao, W. McLean, J. Zhang, A discrete Grönwall inequality with applications to numerical schemes for subdiffusion problems. *SIAM J. Numer. Anal.* 57 (2019), 218–237.

[56] Y. Lin, C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation. *J. Comput. Phys.* 225 (2007), 1533–1552.

[57] Q. Liu, F. Liu, I. Turner, V. Anh, Y. Gu, A RBF meshless approach for modeling a fractal mobile/immobile transport model. *Appl. Math. Comput.* 226 (2014), 336–347.

[58] H. Liu, X. Zheng, H. Wang, H. Fu, Analysis and efficient implementation of ADI finite volume method for Riesz space-fractional diffusion equations in two space dimensions. *Numer. Meth. PDEs* (2020) https://doi.org/10.1002/num.22554.

[59] Y. Liu, W. Rundell, M. Yamamoto, Strong maximum principle for fractional diffusion equations and an application to an inverse source problem. *Fract. Calc. Appl. Anal.* 19 (2016), 888–906.

[60] C. Lorenzo and T. Hartley, Variable order and distributed order fractional operators. *Nonlinear Dynamics* 29 (2002), 57–98.

[61] C. Lubich, Convolution quadrature and discretized operational calculus. I and II. *Numer. Math.* 52 (1988), 129–145 and 413–425.

[62] C. Lubich, I. Sloan, V. Thomée, Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. *Math. Comp.* 65 (1996), 1–17.

[63] Y. Luchko, Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation. *Fract. Calc. Appl. Anal.* 15 (2012), 141–160.

[64] Y. Luchko, M. Yamamoto, On the maximum principle for a time-fractional diffusion equation. *Fract. Calc. Appl. Anal.* 20 (2017), 1131–1145.

[65] C. Lv, C. Xu, Error analysis of a high order method for time-fractional diffusion equations. *SIAM J. Sci. Comput.* 38 (2016), A2699–A2724.

[66] F. Mainardi, M. Raberto, R. Gorenflo, E. Scalas, Fractional calculus and continuous-time finance II: the waiting-time distribution. *Physica A* 287 (2000), 468–481.

[67] Z. Mao, J. Shen, Hermite spectral methods for fractional PDEs in unbounded domains. *SIAM J. Sci. Comput.* 39 (2017), A1928–A1950.

[68] Z. Mao, S. Chen, J. Shen, Efficient and accurate spectral methods using general Jacobi functions for solving Riesz fractional differential equations. *Appl. Numer. Math.* 106 (2016) 165–181.

[69] M.M. Meerschaert and A. Sikorskii, *Stochastic Models for Fractional Calculus*, De Gruyter Studies in Mathematics, 2011.

[70] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, *Phys. Rep.*, 339 (2000), 1-77.

[71] W. McLean, Regularity of solutions to a time-fractional diffusion equation. *The ANZIAM J.* 52 (2010), 123–138.

[72] W. McLean, K. Mustapha, R. Ali, O. Knio, Well-posedness of time-fractional advection-diffusion-reaction equations. *Frac. Calc. Appl. Anal.* 22 (2019), 918–944.

[73] K. Mustapha, FEM for time-fractional diffusion equations, novel optimal error analyses. *Math. Comp.* 87 (2018), 2259–2272.

[74] K. Mustapha, An implicit finite difference time-stepping method for a subdiffusion equation with spatial discretization by finite elements. *IMA J. Numer. Anal.* 31 (2011), 719–739.

[75] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[76] B. Ross, S. Samko, Fractional integration operator of variable order in the Holder spaces $H^{\lambda(x)}$. *Inter. J. Math. & Math. Sci.* 18 (1995), 777–788.

[77] K. Sakamoto and M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.*, 382 (2011), 426–447.

[78] S. Samko, A. Kilbas, and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, London, 1993.

[79] S. Samko and B. Ross, Integration and differentiation to a variable fractional order. *Integral Transforms and Special Functions* 1 (1993), 277–300.

[80] E. Scalas, R. Gorenflo, F. Mainardi, Fractional calculus and continuous-time finance. *Physica A* 284 (2000), 376–384.

[81] R. Schumer, D.A. Benson, M.M. Meerschaert and B. Baeumer, Fractal mobile/immobile solute transport, *Water Resour. Res.*, 39 (2003), 1–12.

[82] J. Shen, T. Tang, and L. Wang, Spectral Methods: Algorithms, Analysis and Applications, Springer Ser. Comput. Math. Vol. 41, Springer, Heidelberg, 2011.

[83] C. Sheng, J. Shen, T. Tang, L. Wang, H. Yuan, Fast Fourier-like mapped Chebyshev spectral-Galerkin methods for PDEs with integral fractional Laplacian in unbounded domains. *SIAM J. Numer. Anal.* (2020) to appear.

[84] M. Stynes, Too much regularity may force too much uniqueness. *Frac. Calc. Appl. Anal.* 19 (2016), 1554–1562.

[85] M. Stynes, E. O'Riordan, and J.L. Gracia, Error analysis of a finite difference method on graded mesh for a time-fractional diffusion equation, *SIAM J. Numer. Anal.*, 55 (2017), 1057–1079.

[86] H. Sun, A. Chang, Y. Zhang, W. Chen, A review on variable-order fractional differential equations: mathematical foundations, physical models, numerical methods and applications. *Fract. Calc. Appl. Anal.* 22 (2019) 27–59.

[87] H. Sun, W. Chen, Y. Chen. Variable-order fractional differential operators in anomalous diffusion modeling. *Physica A: Statistical Mechanics and its Applications*, 388 (2009), 4586–4592.

[88] Z. Sun, C. Ji, R. Du, A new analytical technique of the L-type difference schemes for time fractional mixed sub-diffusion and diffusion-wave equations. *Appl. Math. Lett.* 102 (2020), 106115.

[89] Z. Sun, X. Wu, A fully discrete difference scheme for a diffusion-wave system. *Appl. Numer. Math.* 56 (2006), 193–209.

[90] T. Tang, H. Yuan, T. Zhou, Hermite spectral collocation methods for fractional PDEs in unbounded domains. *Commun. Comput. Phys.* 24 (2018), 1143–1168.

[91] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Lecture Notes in Mathematics 1054, Springer-Verlag, New York, 1984.

[92] S.R. Umarov and S.T. Steinberg, Variable order differential equations with piecewise constant order-function and diffusion with changing modes. *ZAA*, 28 (2009), 131–150.

[93] H. Wang, D. Yang, Wellposedness of variable-coefficient conservative fractional elliptic differential equations. *SIAM J. Numer. Anal.* 51 (2013), 1088–1107.

[94] H. Wang, D. Yang, S. Zhu, Inhomogeneous Dirichlet boundary-value problems of space-fractional diffusion equations and their finite element approximations. *SIAM J. Numer. Anal.* 52 (2014), 1292–1310.

[95] H. Wang, X. Zhang, A high-accuracy preserving spectral Galerkin method for the Dirichlet boundary-value problem of variable-coefficient conservative fractional diffusion equations. *J. Comput. Phys.* 281 (2015), 67–81.

[96] H. Wang and X. Zheng, Wellposedness and regularity of the variable-order time-fractional diffusion equations. *J. Math. Anal. Appl.* 475 (2019), 1778–1802.

[97] H. Wang and X. Zheng, A modified time-fractional diffusion equation and its finite difference method: regularity and error analysis. *Fract. Calc. Appl. Anal.* 22 (2019), 1014–1038.

[98] H. Wang and X. Zheng, Analysis and numerical solution of a nonlinear variable-order fractional differential equation. *Adv. Comput. Math.* 45 (2019), 2647–2675.

[99] X. Wang, S. Xu, S. Zhou, W. Xu, M. Leary, P. Choong, M. Qian, M. Brandt, and Y.M. Xie, Topological design and additive manufacturing of porous metals for bone scaffolds and orthopedic implants. *Biomaterials* 83 (2016), 127–141.

[100] S. Wei, W. Chen, Y. Zhang, H. Wei, R. Garrard, A local radial basis function collocation method to solve the variable-order time fractional diffusion equation in a two-dimensional irregular domain. *Numer. Methods Partial Differ. Eq.* 34 (2018), 1209–1223.

[101] Y. Yan, M. Khan, N. Ford, An analysis of the modified L1 scheme for time-fractional partial differential equations with nonsmooth data. *SIAM J. Numer. Anal.* 56 (2018) 210–227.

[102] Z. Yang, X. Zheng, H. Wang, A variably distributed-order time-fractional diffusion equation: analysis and approximation. *Comput. Meth. Appl. Mech. Engrg.* 367 (2020), 113118.

[103] H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.*, 328 (2007), 1075–1081.

[104] M. Zayernouri, G. Karniadakis, Fractional spectral collocation methods for linear and nonlinear variable order FPDEs. *J. Comput. Phys.* 293 (2015), 312–338.

[105] F. Zeng, Z. Zhang and G. Karniadakis. A generalized spectral collocation method with tunable accuracy for variable-order fractional differential equations. *SIAM J. Sci. Comput.*, 37 (2015), A2710–A2732.

[106] Y. Zhang, D. Benson, M. Meerschaert, E. LaBolle, H. Scheffler, Random walk approximation of fractional-order multiscaling anomalous diffusion. *Phys. Rev. E* 74 (2006), 026706.

[107] X. Zhao, Z. Sun, G. Karniadakis, Second-order approximations for variable order fractional derivatives: algorithms and applications. *J. Comput. Phys.* 293 (2015), 184–200.

[108] X. Zheng, V. J. Ervin, H. Wang, Numerical approximations for the variable coefficient fractional diffusion equations with non-smooth data. *Comp. Meth. Appl. Math.* 20 (2020), 573–589.

[109] X. Zheng, V. J. Ervin, H. Wang, Wellposedness of the two-sided variable-coefficient Caputo flux fractional diffusion equation and error estimate of its spectral approximation. *Appl. Numer. Math.* 153 (2020), 234–247.

[110] X. Zheng, V. J. Ervin, H. Wang, Spectral approximation of a variable coefficient fractional diffusion equation in one space dimension. *Appl. Math. Comput.* 361 (2019), 98–111.

[111] X. Zheng, H. Liu, H. Wang, H. Fu, An efficient finite volume method for the nonlinear distributed-order space-fractional diffusion equations in three space dimensions. *J. Sci. Comput.* 80 (2019), 1395–1418.

[112] X. Zheng, H. Wang, Finite element approximations to variable-order time-fractional diffusion equations and their analysis without regularity assumptions of the solutions. *IMA J. Numer. Anal.* (2020) https://doi.org/10.1093/imanum/draa013.

[113] X. Zheng, H. Wang, An optimal-order numerical approximation to variable-order space-fractional diffusion equations on uniform or graded meshes. *SIAM J. Numer. Anal.* 58 (2020), 330–352.

[114] X. Zheng and H. Wang, An error estimate of a numerical approximation to a hidden-memory variable-order space-time fractional diffusion equation. *SIAM J. Numer. Anal.* 58 (2020), 2492–2514.

[115] X. Zheng and H. Wang, Wellposedness and regularity of a nonlinear variable-order fractional wave equation. *Appl. Math. Lett.* 95 (2019), 29–35.

[116] X. Zheng and H. Wang, Wellposedness and regularity of a variable-order space-time fractional diffusion equation. *Anal. Appl.* 18 (2020), 615–638.

[117] X. Zheng, Z. Zhang, H. Wang, Analysis of a nonlinear variable-order fractional stochastic differential equation. *Appl. Math. Lett.* 107 (2020), 106461.

[118] X. Zheng, H. Wang, H. Fu, Analysis of a physically-relevant variable-order time-fractional reaction-diffusion model with Mittag-Leffler kernel. *Appl. Math. Lett.* (2020) https://doi.org/10.1016/j.aml.2020.106804.

[119] X. Zheng, H. Wang, H. Fu, Well-posedness of fractional differential equations with variable-order Caputo-Fabrizio derivative. *Chaos, Solitons & Fractals* 138 (2020), 109966.

[120] X. Zheng and H. Wang, Variable-order space-fractional diffusion equations and a variable-order modification of constant-order fractional problems. *Appl. Anal.* (2020) https://doi.org/10.1080/00036811.2020.1789596.

[121] X. Zheng and H. Wang, Wellposedness and smoothing properties of history-state-based variable-order time-fractional diffusion equations. *Z. Angew. Math. Phys.* 71 (2020), 34.

[122] X. Zheng, Y. Li, J. Cheng and H. Wang, Inverting the variable fractional order in a variable-order space-fractional diffusion equation with variable diffusivity: analysis and simulation. *J. Inv. Ill-posed Probl.* (2020) to appear.

[123] X. Zheng and H. Wang, Uniquely identifying the variable order of time-fractional partial differential equations on general multi-dimensional domains. *Inverse Probl. Sci. Engrg.* in revision.

[124] X. Zheng and H. Wang, A time-fractional partial differential equation with a space-time dependent hidden-memory variable order: analysis and approximation. Submitted.

[125] A. Zhokh, P. Strizhak, Non-Fickian diffusion of methanol in mesoporous media: geometrical restrictions or adsorption-induced? *J. Chem. Phys.*, 146, 2017, 124704.

[126] P. Zhuang, F. Liu, V. Anh and I. Turner. Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term. *SIAM J. Numer. Anal.*, 47 (2009), 1760–1781.

Appendix A

Permissions to reuse

Contact Request From SIAM

Kelly Thomas <Thomas@siam.org>
Fri 10/16/2020 9:50 AM
To: ZHENG, XIANGCHENG

Dear Mr. Zheng:

SIAM is happy to give permission to reuse material from the articles mentioned below. Please acknowledge the original publications, using the complete bibliographic information.

Sincerely,

Kelly Thomas
Managing Editor
Society for Industrial and Applied Mathematics
3600 Market Street - 6th Floor
Philadelphia, PA 19104
thomas@siam.org / (215) 382-9800 ext. 387

SIAM has made a collection of 150+ epidemiology-related SIAM journal articles, proceedings, and book chapters freely available for one year. Access the collection.

...

ZHENG, XIANGCHENG
Thu 10/15/2020 7:07 PM
To: Karl W. Durst <Durst@siam.org>

Dear Durst,

Thank you for your reply. The two papers are

An Error Estimate of a Numerical Approximation to a Hidden-Memory Variable-Order Space-Time Fractional Diffusion Equation (https://doi.org/10.1137/20M132420X)

An Optimal-Order Numerical Approximation to Variable-order Space-fractional Diffusion Equations on Uniform or Graded Meshes (https://doi.org/10.1137/19M1245621).

OXFORD UNIVERSITY PRESS LICENSE
TERMS AND CONDITIONS

Oct 15, 2020

This Agreement between Mr. Xiangcheng Zheng ("You") and Oxford University Press ("Oxford University Press") consists of your license details and the terms and conditions provided by Oxford University Press and Copyright Clearance Center.

License Number	4930210693664
License date	Oct 15, 2020
Licensed Content Publisher	Oxford University Press
Licensed Content Publication	IMA Journal of Numerical Analysis
Licensed Content Title	Optimal-order error estimates of finite element approximations to variable-order time-fractional diffusion equations without regularity assumptions of the true solutions
Licensed Content Author	Zheng, Xiangcheng; Wang, Hong
Licensed Content Date	Apr 27, 2020
Type of Use	Thesis/Dissertation
Institution name	
Title of your work	Optimal-order error estimates of finite element approximations to variable-order time-fractional diffusion equations without regularity assumptions of the true solutions
Publisher of your work	University of South Carolina
Expected publication date	Oct 2020
Permissions cost	0.00 USD
Value added tax	0.00 USD
Total	**0.00 USD**
Title	Optimal-order error estimates of finite element approximations to variable-order time-fractional diffusion equations without regularity assumptions of the true solutions
Institution name	University of South Carolina
Expected presentation date	Oct 2020
Portions	all tables and figures
Requestor Location	Mr. Xiangcheng Zheng 1642 Enoree Avenue #F COLUMBIA, SC 29205 United States Attn: University of South Crarolina
Publisher Tax ID	GB125506730
Total	**0.00 USD**

Total 0.00 USD

Terms and Conditions

STANDARD TERMS AND CONDITIONS FOR REPRODUCTION OF MATERIAL FROM AN OXFORD UNIVERSITY PRESS JOURNAL

1. Use of the material is restricted to the type of use specified in your order details.
2. This permission covers the use of the material in the English language in the following territory: world. If you have requested additional permission to translate this material, the terms and conditions of this reuse will be set out in clause 12.
3. This permission is limited to the particular use authorized in (1) above and does not allow you to sanction its use elsewhere in any other format other than specified above, nor does it apply to quotations, images, artistic works etc that have been reproduced from other sources which may be part of the material to be used.
4. No alteration, omission or addition is made to the material without our written consent. Permission must be re-cleared with Oxford University Press if/when you decide to reprint.
5. The following credit line appears wherever the material is used: author, title, journal, year, volume, issue number, pagination, by permission of Oxford University Press or the sponsoring society if the journal is a society journal. Where a journal is being

published on behalf of a learned society, the details of that society must be included in the credit line.
6. For the reproduction of a full article from an Oxford University Press journal for whatever purpose, the corresponding author of the material concerned should be informed of the proposed use. Contact details for the corresponding authors of all Oxford University Press journal contact can be found alongside either the abstract or full text of the article concerned, accessible from www.oxfordjournals.org Should there be a problem clearing these rights, please contact journals.permissions@oup.com
7. If the credit line or acknowledgement in our publication indicates that any of the figures, images or photos was reproduced, drawn or modified from an earlier source it will be necessary for you to clear this permission with the original publisher as well. If this permission has not been obtained, please note that this material cannot be included in your publication/photocopies.
8. While you may exercise the rights licensed immediately upon issuance of the license at the end of the licensing process for the transaction, provided that you have disclosed complete and accurate details of your proposed use, no license is finally effective unless and until full payment is received from you (either by Oxford University Press or by Copyright Clearance Center (CCC)) as provided in CCC's Billing and Payment terms and conditions. If full payment is not received on a timely basis, then any license preliminarily granted shall be deemed automatically revoked and shall be void as if never granted. Further, in the event that you breach any of these terms and conditions or any of CCC's Billing and Payment terms and conditions, the license is automatically revoked and shall be void as if never granted. Use of materials as described in a revoked license, as well as any use of the materials beyond the scope of an unrevoked license, may constitute copyright infringement and Oxford University Press reserves the right to take any and all action to protect its copyright in the materials.
9. This license is personal to you and may not be sublicensed, assigned or transferred by you to any other person without Oxford University Press's written permission.
10. Oxford University Press reserves all rights not specifically granted in the combination of (i) the license details provided by you and accepted in the course of this licensing transaction, (ii) these terms and conditions and (iii) CCC's Billing and Payment terms and conditions.
11. You hereby indemnify and agree to hold harmless Oxford University Press and CCC, and their respective officers, directors, employs and agents, from and against any and all claims arising out of your use of the licensed material other than as specifically authorized pursuant to this license.
12. Other Terms and Conditions:

v1.4

Questions? customercare@copyright.com or +1-855-239-3415 (toll free in the US) or +1-978-646-2777.

gapa20:Variable-order space-fractional diffusion equations and a variable-order modification of constant-order fractional problems

Academic UK Non Rightslink <permissionrequest@tandf.co.uk>
Mon 10/19/2020 12:15 AM
To: ZHENG, XIANGCHENG

Dear Xiangcheng Zheng

Xiangcheng Zheng & Hong Wang (2020) Variable-order space-fractional diffusion equations and a variable-order modification of constant-order fractional problems, Applicable Analysis, published online 7th July 2020.

Thank you for your correspondence requesting permission to reproduce the above mentioned material from our Journal in your printed thesis entitled Variable-order fractional partial differential equations: analysis, approximation and inverse problem.

We will be pleased to grant entirely free permission on the condition that you acknowledge the original source of publication and insert a reference to the Journal's web site:
https://www.tandfonline.com/doi/full/10.1080/00036811.2020.1789596

Please note we are unable to grant permission for you to post the **final accepted version** within the 12 month embargo period in your university's repository.

We are happy for you to post your article as it was originally submitted to the journal on the condition that you make the following acknowledgement:

"This is an Author's Original Manuscript of an article published by Taylor & Francis Group in , Applicable Analysis available online at https://www.tandfonline.com/doi/full/10.1080/00036811.2020.1789596

gapa20:Variable-order space-fractional diffusion equations and a variable-order modification of constant-order fractional problems

Academic UK Non Rightslink <permissionrequest@tandf.co.uk>
Mon 10/19/2020 1:36 AM
To: ZHENG, XIANGCHENG

Dear Xiangcheng

You can use parts/all of your **accepted manuscript** in your dissertation.

Due to the 12 month embargo period, you may only post your **Original Manuscript** on your university's repository.

Best wishes

Karin

Information Classification: General